CONCERNING SETS OF POLYNOMIALS ORTHOGONAL SIMULTANEOUSLY ON SEVERAL CIRCLES*

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1. **Introduction.** Starting with the well-known fact that the set of polynomials $\{z^n\}$ is an orthogonal set on every circle $|z|=R$, there has recently been some consideration of the general problem suggested, that of the existence of sets of polynomials in the complex variable which are orthogonal, with respect to suitable norm functions, simultaneously on more than one curve. Terming a set of polynomials $\{p_n(z)\}$ *canonical* on a rectifiable Jordan curve C with respect to the positive continuous norm function $n(z)$ provided it is found by orthogonalizing on C the set $\{z^n\}$ with respect to $n(z)$, and provided the coefficient of z^n in $p_n(z)$ is chosen positive, we list the previous results† which we shall find pertinent to our purpose:

(1) Walsh‡ and Szegö§ have shown, independently and by different methods, that if the same set of polynomials is canonical on two distinct curves then one of the curves is a "level curve" (Kreisbild)‖ in the conformal mapping of the region outside the other curve onto the exterior of a circle, the points at infinity corresponding to each other.

(2) Szegö¶ has exhibited *all* canonical sets of polynomials in the complex variable, each set canonical on *all* level curves of a given family; there are only five essentially different types of such sets.

While this last result is definitive in connection with sets of polynomials canonical simultaneously on a whole family of level curves, the general problem of the existence of sets of polynomials canonical simultaneously on only a *finite* number of curves has not yet been discussed; in the references cited above, Walsh (p. 136) and Szegö (p. 196) both suggest its study. It is the purpose of the present paper

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† A complete list of results on this problem will be found in §1 of Walsh and Merriman, *Note on the simultaneous orthogonality of harmonic polynomials on several curves*, Duke Mathematical Journal, vol. 3 (1937), pp. 279–288.

‡ J. L. Walsh, *Interpolation and Approximation by Rational Functions in the Complex Domain*, American Mathematical Society Colloquium Publications, vol. 20, p. 134, Theorem 11.

§ G. Szegö, *A problem concerning orthogonal polynomials*, Transactions of this Society, vol. 37 (1935), pp. 196–206, Theorem I.

‖ Cf. Walsh, loc. cit., §4.1.

¶ Loc. cit., pp. 197–198.

to consider this problem in the special case that the curves involved are concentric (see (1) above) circles with center at the origin. To state our results briefly, it is shown that if a set of polynomials in the complex variable is a canonical set *simultaneously on two distinct concentric circles* $|z|=R_j>r$, $(j=1, 2)$, *then the set is canonical on all circles* $|z|=R>r$ *concentric with these. Thus, there exist no sets of polynomials in the complex variable canonical simultaneously on only a finite number* (>1) *of concentric circles.*

2. **Preliminaries.** We consider the set of polynomials

$$(1) \qquad p_0(z) = 1,\ p_n(z) = a_0^{(n)} + a_1^{(n)}z + \cdots + a_{n-1}^{(n)}z^{n-1} + z^n, \qquad n = 1, 2, \cdots.$$

The orthogonality criterion used is

$$(2) \qquad \int_{|z|=R} p_k(z)\overline{p_l(z)}n(z)\,|\,dz\,| = 0, \qquad k \neq l,$$

where* $n(z)$ denotes the real, positive, continuous norm function, alterable by multiplication by a positive constant. The function $n(z)$ can be expressed as $n(z)=|D(z)|^2$, where $D(z)$ is analytic and non-vanishing outside a basic circle $|z|=r$, the point $z=\infty$ included. Since $D(z)$ depends only on the polynomials $p_n(z)$, orthogonality on several circles yields one and the same $D(z)$; thus, with allowance for the possible multiplicative constant, we must have $n_j(z)=|D(z)|^2$, $(j=1,2,\cdots)$, on each circle of orthogonality $|z|=R_j$, $(j=1,2,\cdots)$. The whole configuration being studied may be subjected to a linear integral transformation.

The method used is that of determining, from the hypothesis of orthogonality simultaneously on the two distinct circles $|z|=R_j$, $(j=1, 2)$, necessary values of the coefficients $a_i^{(n)}$, $(i=0, 1, \cdots, n-1; n=1, 2, \cdots)$, of the polynomials (1) and a necessary form for a suitable norm function on the two circles. This determination leads to some one of the sets of polynomials which are known to be canonical† on *all* circles $|z|=R>r$:

* The following facts concerning $n(z)$ are digested from Szegö, loc. cit., pp. 197, 201.

† Szegö, loc. cit. We have for our own subsequent purposes recorded slightly more general sets II than those given by Szegö, resulting from a permissible linear integral transformation $z=a^{-1/\alpha}z'$ of his sets; an irrelevant constant has been removed from each $p_n(z)$. Sets II, $a=1$, were exhibited by Szegö (Mathematische Annalen, vol. 79 (1919), pp. 323–339) without mention of orthogonality on more than one curve, and by Walsh (Mémorial des Sciences Mathématiques, no. 73 (1935), p. 43) with mention of orthogonality on all circles $|z|=R>|a|^{1/\alpha}$.

I. *On the circles* $|z|=R>0$, $D(z)=1$, $p_n(z)=z^n$;

II. *On the circles* $|z|=R>|a|^{1/\alpha}$, α *a positive integer*, $a\neq 0$ *a complex constant*, $D(z)=(1-az^{-\alpha})^{-1}$, $p_n(z)=z^n$ *for* $0\leqq n<\alpha$, $p_n(z)=z^{n-\alpha}(z^\alpha-a)$ *for* $n\geqq\alpha$.

The set I is the limiting set of the sets II as $\alpha\to\infty$; the orthogonality of I, even on the circles $|z|=R$, $0<R\leqq|a|^{1/\alpha}$, occurs since in such a limiting process $D(z)$ becomes constant, hence analytic in the whole plane. We thus confine our attention to the case $D(z)\not\equiv 1$, that is, $D(z)$ analytic outside $|z|=r$, the point at infinity included. We shall choose r as the smaller, say R_1, of R_1 and R_2.

3. **Two auxiliary formulas.** We shall use the following form for $n(z)$:

$$n(z)=1+\sum_{j=1}^{\infty}A_jz^j+\sum_{j=1}^{\infty}\overline{A}_j\bar{z}^j. \tag{3}$$

This development (3) is simply the formal Fourier development of the real function $n(z)$ rearranged by means of the equations, valid on $|z|=R$,

$$z^n=R^n(\cos n\theta+i\sin n\theta),\qquad \bar{z}^n=R^n(\cos n\theta-i\sin n\theta),$$

$$\sin n\theta=(z^n-\bar{z}^n)/2iR^n,\qquad \cos n\theta=(z^n+\bar{z}^n)/2R^n.$$

Of course, to obtain the form (3) we have multiplied the preceding development by the reciprocal of its first term, which is positive.

This form of the norm function of sets II is, for z on $|z|=R>|a|^{1/\alpha}$,

$$n(z)=1+\sum_{j=1}^{\infty}R^{-2j\alpha}\bar{a}^jz^{j\alpha}+\sum_{j=1}^{\infty}R^{-2j\alpha}a^j\bar{z}^{j\alpha}. \tag{3'}$$

Use of (3) permits calculation of a set of important auxiliary integrals:

$$\int_{|z|=R}z^k\bar{z}^ln(z)\,|dz|=\begin{cases}2\pi\overline{A}_{k-l}R^{2k+1}, & k>l,\\ 2\pi R^{2k+1}, & k=l,\\ 2\pi A_{l-k}R^{2l+1}, & k<l.\end{cases} \tag{4}$$

For example, if $k>l$,

$$\int_{|z|=R}z^k\bar{z}^ln(z)\,|dz|=R^{2l}\int_{|z|=R}z^{k-l}n(z)\,|dz|$$

$$=R^{2l}\overline{A}_{k-l}\int_{|z|=R}z^{k-l}\bar{z}^{k-l}\,|dz|=R^{2k}\overline{A}_{k-l}\int_{|z|=R}|dz|=2\pi R^{2k+1}\overline{A}_{k-l},$$

by use of the familiar $\int_{|z|=R} z^p \bar{z}^q |dz| = 0$, $p \neq q$. The term-by-term integration is justifiable since the Fourier development of $n(z)$, continuous on $|z| = R$, is uniformly summable (Cesàro) to $n(z)$ on $|z| = R$.

4. **The first steps in an induction.** We proceed to the first steps in an inductive proof. First we have, using (4),

$$\int_{|z|=R} p_1(z)\overline{p_0(z)}n(z)\,|dz| = \int_{|z|=R} (a_0^{(1)} + z)n(z)\,|dz| = 2\pi R[a_0^{(1)} + \overline{A}_1 R^2] = 0$$

for* $R = R_j$, $(j = 1, 2)$. Hence $a_0^{(1)} = -\overline{A}_1 R^2$. Since $a_0^{(1)}$ is the same constant for the two values of R, we must have $\overline{A}_1 R^2 = k_1$, with k_1 a complex constant; thus $A_1 = \bar{k}_1 R^{-2}$, necessarily, on the two circles. We separate these findings into two cases:

$$(5)\qquad \begin{cases} \text{(a)} & k_1 \neq 0;\ a_0^{(1)} = -k_1,\ p_1(z) = z - k_1,\ A_1 = \bar{k}_1 R^{-2}; \\ \text{(b)} & k_1 = 0;\ a_0^{(1)} = 0,\ p_1(z) = z,\ A_1 = 0. \end{cases}$$

We continue with (5a):

$$\int_{|z|=R} p_2(z)\overline{p_0(z)}n(z)\,|dz| = \int_{|z|=R} [a_0^{(2)} + a_1^{(2)} z + z^2]n(z)\,|dz| = 2\pi R[a_0^{(2)} + a_1^{(2)}\overline{A}_1 R^2 + \overline{A}_2 R^4] = 0,$$

so that we have, using (5a),

$$(6)\qquad a_0^{(2)} + k_1 a_1^{(2)} + \overline{A}_2 R^4 = 0.$$

Similarly, with use of (6),

$$\int_{|z|=R} p_2(z)\overline{p_1(z)}n(z)\,|dz| = \int_{|z|=R} \bar{z}[a_0^{(2)} + a_1^{(2)} z + z^2]n(z)\,|dz| = 2\pi R[a_0^{(2)} A_1 R^2 + a_1^{(2)} R^2 + \overline{A}_1 R^4] = 0,$$

which yields the pair of equations

$$a_0^{(2)} k_1 R_j^{-2} + a_1^{(2)} + k_1 = 0, \qquad j = 1, 2.$$

These have the unique solution $a_0^{(2)} = 0$, $a_1^{(2)} = -k_1$, which, with (6), gives $A_2 = \bar{k}_1^2 R_j^{-4}$, $(j = 1, 2)$. We add, then, to (5a) the following information:

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* Henceforth, results involving R will be considered tacitly to be evaluated and valid only for $R = R_j$, $(j = 1, 2)$.

(7) $k_1 \neq 0$; $p_1(z) = z - k_1$, $p_2(z) = z^2 - k_1 z$, $A_1 = \bar{k}_1 R^{-2}$, $A_2 = \bar{k}_1^2 R^{-4}$.

On the basis of (7) we proceed to find cumulatively

$$(8)\qquad \int_{|z|=R} p_3(z)\overline{p_0(z)}n(z)\,|\,dz\,| = 2\pi R\left[a_0^{(3)} + a_1^{(3)}\overline{A}_1R^2 + a_2^{(3)}\overline{A}_2R^4 + \overline{A}_2R^6\right] = 0,$$

$$(9)\qquad \int_{|z|=R} p_3(z)\overline{p_1(z)}n(z)\,|\,dz\,| = 2\pi R\left[a_0^{(3)}A_1R^2 + a_1^{(3)}R^2 + a_2^{(3)}\overline{A}_1R^4 + \overline{A}_2R^6\right] = 0,$$

$$(10)\qquad \int_{|z|=R} p_3(z)\overline{p_2(z)}n(z)\,|\,dz\,| = 2\pi R\left[a_0^{(3)}A_2R^4 + a_1^{(3)}A_1R^4 + a_2^{(3)}R^4 + \overline{A}_1R^6\right] = 0.$$

From (10) and (9) we obtain the four equations

$$(11)\qquad \begin{cases} a_0^{(3)}\bar{k}_1^2 + a_1^{(3)}\bar{k}_1R_j^2 + a_2^{(3)}R_j^4 + k_1R_j^4 = 0, & j = 1, 2, \\ a_0^{(3)}\bar{k}_1 + a_1^{(3)}R_j^2 + a_2^{(3)}k_1R_j^2 + k_1^2R_j^2 = 0, & j = 1, 2. \end{cases}$$

Of these we select the first three, the augmented matrix of which system is

$$M: \left\| \begin{matrix} \bar{k}_1^2 & \bar{k}_1R_1^2 & R_1^4 & -k_1R_1^4 \\ \bar{k}_1^2 & \bar{k}_1R_2^2 & R_2^4 & -k_1R_2^4 \\ \bar{k}_1 & R_1^2 & k_1R_1^2 & -k_1^2R_1^2 \end{matrix} \right\|.$$

We first show that the determinant d_{123} (the subscripts indicate columns of M making up the determinant) is not zero; the method used is chosen to illustrate a later general discussion. We multiply the last row of d_{123} by $\bar{k}_1 \neq 0$ and subtract the first row from the result; then a Laplace development according to the two-rowed determinants of the first two rows yields, aside from an irrelevant non-zero constant multiplier, $d_{123} = (R_2^2 - R_1^2)(|k_1|^2R_1^2 - R_1^4)$. Thus, since $R_1 \neq R_2$, $d_{123} = 0$ when and only when $|k_1| = R_1$; but in this event we can interchange the rôles of R_1 and R_2 in the discussion to obtain a new determinant which, treated in the same manner as the original, can never vanish because now $|k_1| \neq R_2$.

The determinants d_{134} and d_{234} are zero because of the proportionality of the last two columns of M. The determinant $d_{124} = -k_1d_{123}$. Hence the unique solution of the first three equations of (11) is $a_2^{(3)} = -k_1$, $a_0^{(3)} = a_1^{(3)} = 0$; these satisfy the fourth equation of (11). Used in connection with (8) these results produce $A_3 = \bar{k}_1^3R_j^{-6}$,

$(j=1, 2)$. We incorporate these findings with (7) in the necessary forms

$$(12)\qquad \begin{cases} k_1 \neq 0;\ p_1(z) = z - k_1,\ p_2(z) = z^2 - k_1 z,\ p_3(z) = z^3 - k_1 z^2, \\ A_1 = \bar{k}_1 R^{-2},\ A_2 = \bar{k}_1^2 R^{-4},\ A_3 = \bar{k}_1^3 R^{-6}. \end{cases}$$

These are the first three polynomials and A's in set* II, $\alpha=1$, $a=k_1$. Here, of course, for complete identification, in fact for meaning, we must choose $|k_1|^{1/\alpha} = |k_1| < R_1$ (cf. (3′)).

5. **Continuation.** We return to (5b). We now have

$$\int_{|z|=R} p_2(z)\overline{p_0(z)}n(z)\,|\,dz\,| = a_0^{(2)} + \overline{A}_2 R^4 = 0,$$

$$\int_{|z|=R} p_2(z)\overline{p_1(z)}n(z)\,|\,dz\,| = a_1^{(2)} R^2 = 0.$$

Hence $a_1^{(2)}=0$, $a_0^{(2)} = -\overline{A}_2 R^4 = -k_2$, k_2 a complex constant. The following two possibilities are then to be added to (5b):

$$(13)\qquad \begin{cases} \text{(a)}\ \ k_1 = 0,\ k_2 \neq 0;\ p_1(z) = z,\ p_2(z) = z^2 - k_2,\ A_1 = 0,\ A_2 = \bar{k}_2 R^{-4}; \\ \text{(b)}\ \ k_1 = k_2 = 0;\quad p_1(z) = z,\ p_2(z) = z^2,\qquad A_1 = A_2 = 0. \end{cases}$$

Pursuing first (13a) we have the following replacements for (8), (9), and (10):

$$(14)\qquad a_0^{(3)} + a_2^{(3)} k_2 + \overline{A}_3 R_j^6 = 0, \qquad j = 1, 2,$$

$$(15)\qquad a_1^{(3)} R_j^2 + k_2 R_j^2 = 0, \qquad j = 1, 2,$$

$$(16)\qquad a_0^{(3)} \bar{k}_2 + a_2^{(3)} R_j^4 = 0, \qquad j = 1, 2.$$

From the last four equations we choose the system formed by (16) and the first equation of (15); its augmented matrix is

$$\left\| \begin{matrix} \bar{k}_2 & 0 & R_1^4 & 0 \\ \bar{k}_2 & 0 & R_2^4 & 0 \\ 0 & R_1^2 & 0 & -k_2 R_1^2 \end{matrix} \right\|.$$

Again $d_{123} \neq 0$, but d_{124} and d_{234} are zero, while $d_{134} = k_2 d_{123}$. Thus $a_0^{(3)} = a_2^{(3)} = 0$, $a_1^{(3)} = -k_2$, uniquely, and (14) gives $A_3 = 0$. We continue (13a) as

* That is, the polynomials and A's with subscripts 1, 2, 3; here and later we omit $p_0(z)$ and A_0 from the list since they are unity in all cases.

$$(17)\qquad \begin{aligned} &k_1 = 0,\ k_2 \neq 0;\ p_1(z) = z,\ p_2(z) = z^2 - k_2,\ p_3(z) = z^3 - k_2 z,\\ &A_1 = A_3 = 0,\ A_2 = \bar{k}_2 R^{-4}. \end{aligned}$$

These are the first three polynomials and A's of set II, $\alpha=2$, $a=k_2$, $|k_2|^{1/\alpha}=|k_2|^{1/2}<R_1$.

6. **Continuation.** Returning to (13b), we now have the equations $a_0^{(3)}+\overline{A}_3 R_j^6=0$, $a_1^{(3)}R_j^2=0$, $a_2^{(3)}R_j^4=0$, $(j=1, 2)$, which have the immediate unique solution $a_1^{(3)}=a_2^{(3)}=0$, $a_0^{(3)}=-\overline{A}_3R^6=-k_3$, so that (13b) is continued as

$$(18)\qquad \begin{cases} \text{(a)}\quad k_1 = k_2 = 0,\ k_3 \neq 0;\ p_1 = z,\ p_2 = z^2,\ p_3 = z^3 - k_3;\\ \qquad\qquad\qquad\qquad\qquad\qquad A_1 = A_2 = 0,\ A_3 = k_3 R^{-6};\\ \text{(b)}\quad k_1 = k_2 = k_3 = 0;\qquad p_1 = z,\ p_2 = z^2,\ p_3 = z^3;\\ \qquad\qquad\qquad\qquad\qquad\qquad A_1 = A_2 = A_3 = 0. \end{cases}$$

In (18a) *we have the first three polynomials and A's in set* II, $\alpha=3$, $a=k_3$, $|k_3|^{1/\alpha}=|k_3|^{1/3}<R_1$. *In* (18b) *we have the first three polynomials and A's in all* sets* II, $\alpha>3$.

7. **The general step.** For the general step in the inductive proof we follow the lead given by (12), (17), and (18) and the italicized statements following them, and assume that, on the hypothesis of orthogonality on the two distinct circles $|z|=R_j$, $(j=1, 2)$, the first m polynomials of (1) have been found, on these two circles, to be necessarily identical with the first m polynomials in some specific one, say (σ), of the sets II, $\alpha\leqq m$, or with the first m polynomials in all sets II, $\alpha>m$, while the norm function has been determined to the same extent of coincidence. We should then expect to show that the hypothesis implies that the first $m+1$ polynomials of (1) must be, on the two circles, identical with the first $m+1$ polynomials in the specified set (σ) of II, $\alpha\leqq m$, or in the set II, $\alpha=m+1$, or in all the sets II, $\alpha>m+1$,† while the A_{m+1} obtained in the process is the one anticipated. These expectations are realized; we outline the proof.

We first evaluate the integrals

$$\int_{|z|=R} p_{m+1}(z)\overline{p_i(z)}n(z)\,|\,dz\,| = 0, \qquad i = 0, 1, 2, \cdots, m.$$

* The first m polynomials and A's in *all* sets II, $\alpha>m$, are the same.

† The last two possibilities compose the "split" case analogous to (18) which, we recall, may lead in the limit, $\alpha\to\infty$, to case I.

Here $p_{m+1}(z)$ is the $(m+1)$th polynomial in (1), $p_i(z)$, $(i=1, 2, \cdots, m)$, are the first m polynomials after $p_0(z)=1$ in some arbitrary set (σ) of II, $\alpha \leqq m$, or in all sets II, $\alpha > m$, and

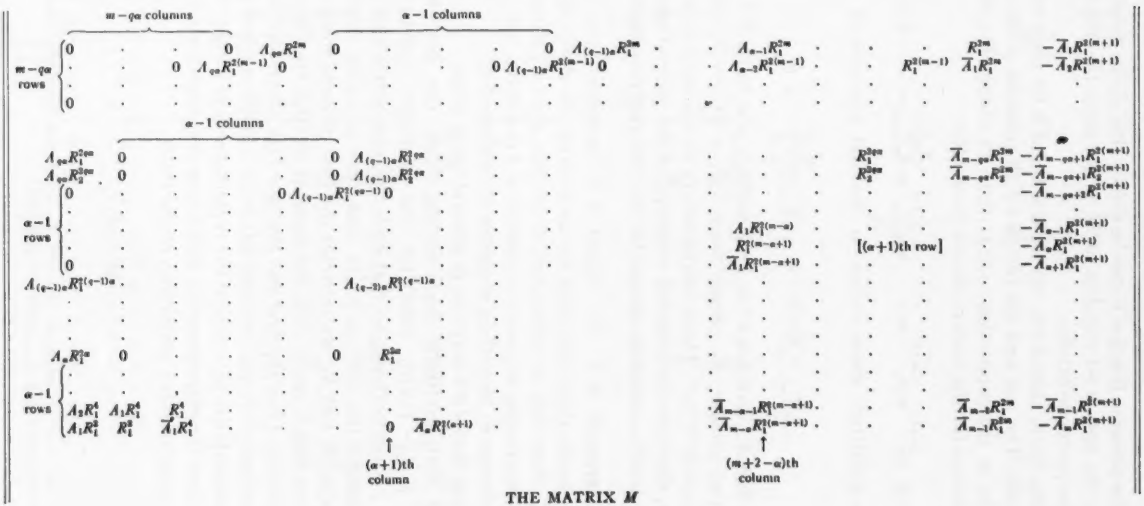
$$
\begin{aligned}
n(z) = 1 + 0 + \cdots + 0 + R^{-2\alpha}\bar{a}z^{\alpha} + \cdots + R^{-2q\alpha}\bar{a}^{q}z^{q\alpha} \\
+ 0 + \cdots + 0 + R^{-2\alpha}a\bar{z}^{\alpha} + \cdots + R^{-2q\alpha}a^{q}\bar{z}^{q\alpha} \\
+ \sum_{j=m+1}^{\infty} A_j z^j + \sum_{j=m+1}^{\infty} \overline{A}_j \bar{z}^j;
\end{aligned} \tag{19}
$$

in (19) $q\alpha$ is the largest integral multiple of α not greater than m, and all A's with subscripts between multiples of α are zero; also we assume, for meaning, $|a|^{1/\alpha} < R_1$. The cumulative results of the integrations are the $2(m+1)$ equations (we have retained the vanishing A's for symmetry of writing):

$$
\left\{
\begin{array}{l}
a_0^{(m+1)} + a_1^{(m+1)}\overline{A}_1 R_j^2 + a_2^{(m+1)}\overline{A}_2 R_j^4 + \cdots + \overline{A}_{m+1} R_j^{2(m+1)} = 0, \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad j = 1, 2, \\
a_0^{(m+1)} A_1 R_j^2 + a_1^{(m+1)} R_j^2 + a_2^{(m+1)}\overline{A}_1 R_j^4 + \cdots + \overline{A}_m R_j^{2(m+1)} = 0, \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad j = 1, 2, \\
a_0^{(m+1)} A_2 R_j^4 + a_1^{(m+1)} A_1 R_j^4 + a_2^{(m+1)} R_j^4 + \cdots + \overline{A}_{m-1} R_j^{2(m+1)} = 0, \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad j = 1, 2, \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
a_0^{(m+1)} A_m R_j^{2m} + a_1^{(m+1)} A_{m-1} R_j^{2m} + \cdots + a_m^{(m+1)} R_j^{2m} \\
\qquad\qquad\qquad\qquad\qquad\qquad + \overline{A}_1 R_j^{2(m+1)} = 0, \qquad j = 1, 2.
\end{array}
\right. \tag{20}
$$

Omitting the first pair of equations in (20) since it contains the unknown $\overline{A}_{m+1}$ as well as the unknowns $a_i^{(m+1)}$, $(i=0, 1, \cdots, m)$, and reading the other pairs in reverse order, we select the pair whose leading term is $a_0^{(m+1)} A_{q\alpha} R_j^{2q\alpha}$, $(j=1, 2)$, and the first one of each other pair. Thus we have a system of $m+1$ linear non-homogeneous equations in the $m+1$ unknowns $a_i^{(m+1)}$, $(i=0, 1, \cdots, m)$. The augmented matrix, M, of this system is displayed in skeleton form on page 65. In the left-hand part of M we have indicated the diagonal panels of zeros arising from the vanishing A's in $n(z)$; in the right-hand part the zeros have not been inserted, the interest here being on the subscripts of the A's in the final columns.

Our general argument is valid unless $\alpha > m$, when the first and last columns of M consist exclusively of zeros; this special case we post-


m − qα columns
α − 1 columns
m − qα rows
α − 1 columns
α − 1 rows
α − 1 rows
(α+1)th column
(m+2−α)th column
[(α+1)th row]

THE MATRIX M

pone for the present. If $\alpha \leqq m$ it can be shown that d, the determinant formed by the first $m+1$ columns of M, is not zero; the proof of this fact we also postpone briefly.

Assuming, then, that $d \neq 0$, we replace the A's by their values as given in (19). The last and the $(m-\alpha+2)$th columns of M are then seen* to be in the proportion $-a:1$; thus we obtain the following unique solution for the system chosen from (20):

$$a_{m+1-\alpha}^{(m+1)} = -a, \quad a_0^{(m+1)} = a_1^{(m+1)} = \cdots = a_{m-\alpha}^{(m+1)} = a_{m-\alpha+2}^{(m+1)} = \cdots = a_m^{(m+1)} = 0.$$

When we substitute these values in the first equation of (20), it becomes

$$-a\overline{A}_{m+1-\alpha}R^{2(m+1-\alpha)} + \overline{A}_{m+1}R^{2(m+1)} = 0. \tag{21}$$

Hence we find $\overline{A}_{m+1} = 0$ if $m+1$ is not a multiple of α, but if $m+1 = q'\alpha$, q' a positive integer, we obtain from (21) $a \cdot a^{q'-1} = \overline{A}_{m+1}R^{2q'\alpha}$, or $\overline{A}_{m+1} = \overline{A}_{q'\alpha} = R^{-2q'\alpha}a^{q'}$. These results satisfy all the equations of (20). Moreover, they are the expected results; and the general step in the induction proof is complete except for the two postponements.

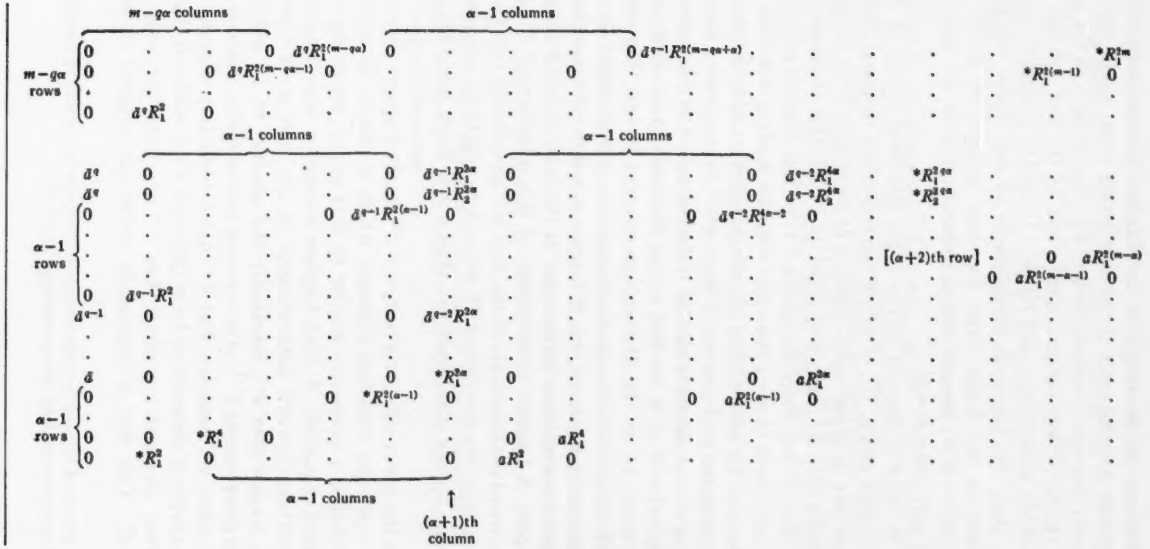
8. **Non-zeroness of d.** We exhibit d in skeleton form on page 67; the elements there indicated by an asterisk we shall call "stop-elements"; they may be considered to involve $\bar{a}^0$ or a^0, stopping the preceding descending sequence of powers of $\bar{a}$ in a given row or starting the subsequent ascending sequence of powers of a.

The proof that $d \neq 0$ involves in general three steps.

(i) We first multiply the (reading from the bottom) αth, 2αth, $\cdots$, $[(q-1)\alpha]$th rows by the non-zero numbers $\bar{a}^{q-1}$, $\bar{a}^{q-2}$, $\cdots$, $\bar{a}$, respectively, so that each row now starts with 0 or $\bar{a}^q$. Leaving intact the $(m-q\alpha+1)$th and $(m-q\alpha+2)$th rows, those symmetric in R_1 and R_2, we subtract the first of these from each remaining row which starts with the element $\bar{a}^q$ (this step is unnecessary if $\alpha > m/2$). Then the first and the $(\alpha+1)$th columns have zeros everywhere except in the two rows left intact; and in each of the new rows all elements are zero except the replacements of former non-zero elements after the stop-elements, which now are, aside from an irrelevant non-zero constant multiplier, of the form

$$|a|^r R_1^s - R_1^t, \tag{22}$$

* In the last row, for example, $\bar{A}_m$ and $\bar{A}_{m-\alpha}$ are zero together unless $m = q\alpha$; then $\bar{A}_m R_1^{2(m+1)} = a^q R_1^2$ and $\bar{A}_{m-\alpha}R_1^{2(m-\alpha+1)} = a^{q-1}R_1^2$.



THE DETERMINANT d
(* indicates "stop-element")

where $t=\alpha r+s$. We develop d in the Laplace manner according to the two-rowed determinants of the two intact rows; again omitting an irrelevant non-zero constant factor, we thus reveal d as the product of U, the two-rowed determinant formed by the first and $(\alpha+1)$th columns of the intact rows, and its $(m-1)$-square algebraic complement d', since the algebraic complement of any other two-rowed determinant of the intact rows has one column of zeros. But $U=\bar{a}^{2q-1}(R_2^{2\alpha}-R_1^{2\alpha})$ cannot vanish because $R_1\neq R_2$, so that $d=0$ *when and only when* $d'=0$.

(ii) If $m-q\alpha\neq 0$, there is in the upper left-hand corner of d' an $(m-q\alpha)$-square array, U', of which the secondary diagonal, reading from lower left to upper right, consists of successive even powers of R_1, beginning with R_1^2, each multiplied by $\bar{a}^q$; all other elements of U' are zero, thus $U'\neq 0$. Since $m-q\alpha\leqq\alpha-1$, we can produce repetitions of U' in each block of $\alpha-1$ rows and columns reading down the left-hand side of d' by multiplying by suitable powers of $\bar{a}$ the rows repeating the pattern of powers of R_1 in U'. Leaving intact the rows containing U', we subtract each of them from each of the later rows now identical with it in its first $m-q\alpha$ elements. As a result, every element below U' in the first $m-q\alpha$ columns is zero; beyond the $(m-q\alpha)$th column zeros occur consistently in the altered rows until the stop-element is passed, while former non-zero elements beyond that element are replaced by elements of the form (22) and all others are still zero. A Laplace development of this transformed d' by the $(m-q\alpha)$-rowed determinants of the first $m-q\alpha$ columns reveals that d' is essentially the product of $U'\neq 0$ and its algebraic complement d'', of $q\alpha-1$ rows and columns; thus $d'=0$ *when and only when* $d''=0$.

(iii) In the upper left-hand corner of d'' occurs a square array, U'', of $\alpha-1-(m-q\alpha)$ rows and columns, which is like U' in structure. We now follow a procedure similar to that in (ii), replacing U' by U'' in that argument. A final Laplace expansion according to the $[\alpha-1-(m-q\alpha)]$-rowed determinants of the first $\alpha-1-(m-q\alpha)$ columns reveals that d'', essentially the product of $U''\neq 0$ by its algebraic complement d''', of $m-\alpha$ rows and columns, *vanishes when and only when* d''' *vanishes*. But d''' has a secondary diagonal consisting entirely of elements of form (22) with $r=2$, while all elements above that diagonal are zero; thus $d'''=0$ *when and only when* $|a|^{1/\alpha}=R_1$. This last is impossible under our original choice* of

* As a matter of fact, we can also circumvent the case $|a|^{1/\alpha}=R_1$ by the argument employed in connection with the determinant d_{123} in §4.

$|a|^{1/\alpha}<R_1$; hence $d'''\neq 0$. Thus $d\neq 0$ as was to be proved.*

9. **The case $\alpha>m$.** We have only to discuss the case $\alpha>m$. Here $d=0$ since the first column is all zeros, so the preceding argument fails. But in this case the A_j, $(j=1, 2, \cdots, m)$, are all zero and the equations (20) after the first pair yield $a_j^{(m+1)}=0$, $(j=1, 2, \cdots, m)$. The first pair, now

$$a_0^{(m+1)} + \overline{A}_{m+1}R_j^{2(m+1)} = 0, \qquad j = 1, 2,$$

produces in the usual way $\overline{A}_{m+1}=k_{m+1}R_j^{-2(m+1)}$, k_{m+1} a complex constant, $a_0^{(m+1)}=-k_{m+1}$. Since k_{m+1} may, or may not, be zero, we have the "split" case previously mentioned; in the former event we have the first $m+1$ polynomials and A's in II, $\alpha=m+1$, in the latter event we have the first $m+1$ polynomials and A's in *all* sets II, $\alpha>m+1$.

10. **Conclusion.** The general step in the induction is now established and the proof of the form of (1) and (3) is complete. We have proved that if (1) is a canonical set simultaneously on *two* distinct circles $|z|=R_j$, $(j=1, 2; R_1<R_2)$, then *on those circles* the set (1) must be identical with some set, (σ), of II, $|a|^{1/\alpha}<R_1$, or with I. But then, since (σ) and I are canonical sets on *all* circles $|z|=R>|a|^{1/\alpha}$, we have established the fact that (1) *is a canonical set simultaneously on all circles* $|z|=R>|a|^{1/\alpha}$ *if it is a canonical set on two such circles.*

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* The first step in this proof is unnecessary if $\alpha>m/2$, the second if $m-q\alpha=0$, the third if $m-q\alpha=\alpha-1$. The cases $m=14$, $\alpha=4, \cdots, 8$ are apt illustrations of all the various possibilities.

THE RESOLVENT OF A CLOSED TRANSFORMATION

A. E. TAYLOR

1. **Introduction.** In reading over Chapter IV of Stone's book, *Linear Transformations in Hilbert Space*, I was impressed by the fact that a number of the results obtained are valid for any complex Banach space. This generality does not always appear at once evident, and it may be worth while to explain briefly.

The most interesting and important fact which underlies the material is that the resolvent of a closed distributive* transformation depends *analytically* on a parameter λ. This dependence is made precise in Stone's work with the aid of the inner product of Hilbert space; but this is not necessary, for it is known that the fundamental portions of the classical theory of analytic functions remain valid in complex Banach spaces.† In particular, Liouville's theorem admits a valid generalization. Thus we are able to prove that the spectrum of a (continuous) linear transformation whose domain is the whole space E is not empty. We shall now turn to the details.

2. **Preliminaries.** We use E to denote a complex Banach space; T will denote a distributive (additive, homogeneous) transformation, with domain and range both in E. We then write $T_\lambda = T - \lambda I$, and T_λ^{-1} will denote the inverse of T_λ when it exists. Here λ is a complex number, and I the identity transformation. We recall that a transformation admits an inverse if and only if it sets up a one to one correspondence between its domain and its range. When T is distributive, the necessary and sufficient condition that T_λ^{-1} exist is that $T_\lambda f = 0$ imply $f = 0$. The set of values of λ for which T_λ^{-1} is linear, with domain everywhere dense in E, is called the resolvent set of T. All other values of λ constitute the spectrum of T.

3. **Discussion of the resolvent.** We prove the following theorem:

THEOREM 1. *If $T_{\lambda_0}^{-1}$ exists and is linear, then T_λ^{-1} exists and is linear for each λ such that $|\lambda - \lambda_0| < 1/M_{\lambda_0}$, where M_{λ_0} is the modulus of $T_{\lambda_0}^{-1}$.*

* We use *distributive* where Stone uses *linear*, preferring to reserve the latter term for continuous distributive transformations. For the definition of a closed transformation see Stone, loc. cit., p. 38.

† For the independent variable a complex number this was pointed out by Wiener, Fundamenta Mathematicae, vol. 4 (1923). For the general case see A. E. Taylor, Comptes Rendus, vol. 203 (1936), pp. 1228–1230, and a forthcoming paper in the Annali della Reale Scuola Normale di Pisa; also L. M. Graves, this Bulletin, vol. 41 (1935), pp. 651–653.

PROOF. Let f be the domain of T. Then $T_\lambda f = T_{\lambda_0} f - (\lambda - \lambda_0) f$. But since $T_{\lambda_0}^{-1}$ is linear, $\|f\| \leqq M_{\lambda_0} \|T_{\lambda_0} f\|$, and hence if $|\lambda - \lambda_0| < 1/M_{\lambda_0}$, $T_\lambda f = 0$ implies $f = 0$, so that T_λ^{-1} exists. Also, since

$$\|Tf - \lambda f\| \geqq \|T_{\lambda_0} f\| - |\lambda - \lambda_0| \, \|f\| \geqq \left(\frac{1}{M_{\lambda_0}} - |\lambda - \lambda_0| \right) \|f\|,$$

we conclude that T_λ^{-1} is bounded, and hence linear. Its modulus does not exceed $M_{\lambda_0}/(1 - M_{\lambda_0} |\lambda - \lambda_0|)$.

THEOREM 2. *If T is closed, and if its resolvent set is not empty, the domain of T_λ^{-1} is the whole space E when λ is in the resolvent set.*

The proof of this offers no difficulties, and we omit it. The family of linear transformations T_λ^{-1} is called the resolvent. We denote it by R_λ.

The next theorem is a direct carry over from Stone (Theorem 4.10, p. 137). The proof given by Stone is valid in the present case, and the reader is referred to it.

THEOREM 3. *If the resolvent R_λ of a closed distributive transformation T exists, then $R_\lambda - R_\mu \equiv (\lambda - \mu) R_\lambda R_\mu$ throughout E for each pair of values λ, μ in the resolvent set, and $R_\lambda f = 0$ implies $f = 0$.*

Conversely, if X_λ is a family of linear transformations with domain E, defined for each λ in a set Σ, such that

$$X_\lambda - X_\mu \equiv (\lambda - \mu) X_\lambda X_\mu \tag{1}$$

for each λ, μ in Σ; and if $X_\lambda f = 0$ implies $f = 0$ for at least one λ in Σ, then there exists a unique closed, distributive transformation T whose resolvent exists and coincides with X_λ for every λ in Σ.

The functional equation (1) is striking. It suggests at once a "law of the mean" for $X_\lambda f$, and in a sense is just that. We shall consider some further conclusions which may be drawn from (1) under suitable hypotheses.

THEOREM 4. *Let X_λ be a linear transformation with domain E for each λ in an open set Σ of the complex plane; further, let $X_\lambda f$ be continuous in Σ for each f in E, and suppose that the functional equation (1) is satisfied when λ, μ are in Σ. Then $X_\lambda f$ is analytic* in Σ for each f in E, and in the neighborhood of each point of Σ admits an expansion in the form*

* In the sense indicated by Wiener. See the references in an earlier footnote.

$$X_\lambda f = \sum_{n=0}^{\infty} (\lambda - \lambda_0)^n X_{\lambda_0}^{n+1} f, \tag{2}$$

where λ_0 is in Σ and $|\lambda - \lambda_0|$ is sufficiently small.

PROOF. We first prove that $X_\lambda f$ admits a derivative with respect to λ at each point of Σ. The form of (1) suggests that this derivative is $X_\lambda^2 f$; this surmise is verified by the inequality

$$\left\| \frac{X_\lambda f - X_{\lambda_0} f}{\lambda - \lambda_0} - X_{\lambda_0}^2 f \right\| = \| X_{\lambda_0}(X_\lambda f - X_{\lambda_0} f) \| \leqq M_{\lambda_0} \| X_\lambda f - X_{\lambda_0} f \|,$$

where M_{λ_0} is the modulus of X_{λ_0}, since $X_\lambda f$ is continuous with respect to λ (it is easily seen that $X_\lambda X_\mu \equiv X_\mu X_\lambda$). It follows from the general theory of abstract-valued analytic functions that $X_\lambda f$ is analytic in Σ. In particular, it admits derivatives of all orders in Σ, and is expansible in a Taylor series. In order to show that this expansion about a point $\lambda = \lambda_0$ has the form (2), we shall prove that the derivatives of $X_\lambda f$ are given by the formula

$$[X_\lambda f]^{(n)} \equiv \frac{d^n}{d\lambda^n} [X_\lambda f] = n! X_\lambda^{n+1} f. \tag{3}$$

This has already been established for $n = 1$. We proceed by induction, assuming the truth of (3) with n replaced by $n-1$. Then

$$\begin{aligned}
&\left\| \frac{[X_\lambda f]^{(n-1)} - [X_{\lambda_0} f]^{(n-1)}}{\lambda - \lambda_0} - n! X_{\lambda_0}^{n+1} f \right\| \\
&= \left\| (n-1)! \frac{X_\lambda^n f - X_{\lambda_0}^n f}{\lambda - \lambda_0} - n! X_{\lambda_0}^{n+1} f \right\| \\
&= \left\| (n-1)! \frac{X_\lambda - X_{\lambda_0}}{\lambda - \lambda_0} (X_\lambda^{n-1} + X_\lambda^{n-2} X_{\lambda_0} + \cdots + X_{\lambda_0}^{n-1}) f - n! X_{\lambda_0}^{n+1} f \right\| \\
&= \| (n-1)! X_{\lambda_0} X_\lambda (X_\lambda^{n-1} + X_\lambda^{n-2} X_{\lambda_0} + \cdots + X_{\lambda_0}^{n-1}) f - n! X_{\lambda_0}^{n+1} f \| \\
&\leqq (n-1)! M_{\lambda_0} \| (X_\lambda^n + X_\lambda^{n-1} X_{\lambda_0} + \cdots + X_\lambda X_{\lambda_0}^{n-1}) f - n X_{\lambda_0}^n f \|.
\end{aligned}$$

But the expression inside the last norm sign tends to zero as $\lambda \to \lambda_0$, by virtue of the continuity of X_λ, and so the induction is completed. The continuity of the iterates of X_λ, and therefore of the expression inside the norm, is deduced from the fact that $X_\lambda f$, being linear in f, is continuous in λ and f *together*.*

* This is a theorem of Kerner, Studia Mathematica, vol. 3 (1931), p. 159. Formula (3) could also be established by use of some theorems about Fréchet differentials. For the above direct proof I am indebted to the referee.

From (3) and the fact that Σ is an open set we can infer the validity of (2) for some region $|\lambda-\lambda_0|<\rho$. This completes the proof of Theorem 4.

However, it is apparent that (2) converges and defines a linear transformation with domain E whenever $|\lambda-\lambda_0|<1/M_{\lambda_0}$. If this range of values was not originally included in Σ, the domain of definition of X_λ may be extended, and it is easy to see that (1) will continue to be satisfied.

The foregoing considerations suggest at once the nature of the resolvent of T.

THEOREM 5. *Let T be a closed, distributive transformation whose resolvent exists. Then the resolvent set Σ is an open set, and the resolvent $R_\lambda f$ is an analytic function of λ in Σ. If M_λ is the modulus of R_λ, Σ is such that with λ_0 it contains all points λ such that $|\lambda-\lambda_0|<1/M_{\lambda_0}$. R_λ is then given by the expansion*

$$R_\lambda f=\sum_{n=0}^{\infty}(\lambda-\lambda_0)^n R_{\lambda_0}^{n+1} f \tag{4}$$

and its modulus satisfies the inequality

$$M_\lambda \leqq \frac{M_{\lambda_0}}{1-|\lambda-\lambda_0|\,M_{\lambda_0}}. \tag{5}$$

PROOF. Since Σ is non-empty, it contains a point λ_0, and R_{λ_0} is a linear transformation with domain E (Theorem 2). Let us define a family of transformations $X_\lambda f$ by the series on the right in equation (4). It is clear that $X_\lambda f$ will be linear, with domain E, when $|\lambda-\lambda_0|<1/M_{\lambda_0}$. Next we identify X_λ with T_λ^{-1} when $|\lambda-\lambda_0|<1/M_{\lambda_0}$. The calculations are identical with those in Stone, pages 140–141, and will be omitted. By definition, then, the resolvent R_λ of T exists and coincides with X_λ when $|\lambda-\lambda_0|<1/M_{\lambda_0}$. The resolvent set is accordingly open. Equation (4) now gives $R_\lambda f$ as a power series with abstract coefficients; it is therefore analytic as a function of λ. This result may be obtained without further resort to the theory of abstract power series by appealing to Theorem 4, since $R_\lambda f$ is now evidently continuous as a function of λ, and satisfies the functional equation (1), by Theorem 3. The inequality (5) has already been noted in Theorem 1. Theorem 5 is thus proved.

4. **The resolvent of a linear transformation.** We state now another theorem:

THEOREM 6. *If T is linear, with domain E and modulus C, the resolvent set contains all λ such that $|\lambda| > C$, and for these values*

$$R_\lambda f = -\sum_{n=1}^{\infty} \frac{1}{\lambda^n} T^{n-1} f. \tag{6}$$

Thus all the singularities of $R_\lambda f$ lie in the finite part of the plane. In particular $R_\lambda f$ has at least one singularity, that is, the spectrum of T is not empty.

The proof of (6) is well known. The last assertion of the theorem follows from Liouville's theorem (for abstract analytic functions), since from (6) we have

$$\|R_\lambda f\| \leqq \frac{\|f\|}{|\lambda| - C}, \qquad |\lambda| > C.$$

If the spectrum of T were empty, $R_\lambda f$ (f fixed) would be analytic over the entire plane and finite at infinity. Then $R_\lambda f$ would be a constant, with value 0, for all f, since $\lim_{|\lambda| \to \infty} \|R_\lambda f\| = 0$. This is impossible. (We exclude, of course, the trivial case where E consists of the zero element alone.)

It would be interesting to know other relationships between the nature of T and the singularities of its resolvent.

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LINEAR OPERATIONS ON FUNCTIONS OF BOUNDED VARIATION

T. H. HILDEBRANDT

The object of this note is to give a form for the most general linear continuous operation on the space of functions of bounded variation on a finite interval, say $0 \leqq x \leqq 1$, the norm of the space being the total variation.

This form is obtained by setting up an equivalent space. For this purpose let $\mathfrak{I}$ be the class of elements I consisting of any finite number of non-overlapping intervals $i_1, \cdots, i_n$ of the interval $(0, 1)$. If (x_p, y_p) are endpoints of i_p, define the function of interval sets $\beta(I) = \sum_{p=1}^{n} [\alpha(y_p) - \alpha(x_p)]$ corresponding to the function $\alpha(x)$ of bounded variation. Then $\beta(I)$ is a bounded function on $\mathfrak{I}$. Define $\|\beta\|$ in the usual way as the least upper bound of $|\beta(I)|$ for I on $\mathfrak{I}$. Then the space $\mathfrak{B}$ of additive set functions β thus normed is equivalent to the space $\mathfrak{A}$ of functions $\alpha(x)$ of bounded variation with $\|\alpha\| = V\alpha = \int_0^1 |d\alpha|$,* for obviously $\|\beta\| \leqq \|\alpha\| \leqq 2\|\beta\|$. Further, if α_1 corresponds to β_1 and α_2 to β_2, then $\beta_1 + \beta_2$ corresponds to $\alpha_1 + \alpha_2$ and $c\beta$ to $c\alpha$, and conversely.

It is now an easy matter to determine the most general linear functional operation on the space $\mathfrak{B}$. Following the lines of reasoning of my paper *On bounded linear functional operations*,† one finds that for any linear continuous operation L on the space $\mathfrak{B}$ there exists an additive function γ of sets E of elements I, such that $L(\beta) = \int \beta d\gamma$, the integral being of the L or S type as defined in the paper quoted, and extended over the class of all subsets of elements of $\mathfrak{I}$. Because of the relationship between the functions β and α this gives the most general linear operation in the space $\mathfrak{A}$.

It might be noted that a similar reasoning applies to the set of interval functions $\alpha(i)$ where $\sum_{p=1}^{n} \alpha(i_p) = \beta(I)$ is a bounded function on $\mathfrak{I}$; or, more generally, that a similar result holds in the space of bounded functions on a general range, with norm the least upper bound of the absolute value of the function on the range.

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* Note that in the space $\mathfrak{A}$ two functions for which $V(\alpha_1 - \alpha_2) = \int |d(\alpha_1 - \alpha_2)| = 0$ are regarded as equivalent. To obtain uniqueness, the condition $\alpha(0) = 0$ can be added. If we wish that $\|\alpha\| = 0$ imply $\alpha = 0$ for all x, we may choose $\|\alpha\| = |\alpha(0)| + V\alpha$. The space $\mathfrak{B}_1$ corresponding is defined by $\beta_1(I) = \alpha(0) + \sum_{p=1}^{n} [\alpha(y_p) - \alpha(x_p)] = \alpha(0) + \beta(I)$ and $\|\beta_1(I)\| = |\alpha(0)| + \|\beta(I)\|$. Reasoning similar to the above can be carried through in this case also.

† Transactions of this Society, vol. 36 (1934), pp. 868–875.

A NOTE ON LINEAR TOPOLOGICAL SPACES*

D. H. HYERS

A space T is called a linear topological space if (1) T forms a linear† space under operations $x+y$ and αx, where $x,y\epsilon T$ and α is a real number, (2) T is a Hausdorff topological space,‡ (3) the fundamental operations $x+y$ and αx are continuous with respect to the Hausdorff topology. The study§ of such spaces was begun by A. Kolmogoroff (cf. [4]. Kolmogoroff's definition of a linear topological space is equivalent to that just given). Kolmogoroff calls a set $S\subset T$ bounded, if for any sequence $x_\nu\epsilon S$ and any real sequence α_ν converging to 0 we have $\lim_{\nu\to\infty}\alpha_\nu x_\nu=\theta$, where θ is the zero element of T. He then shows that a linear topological space T reduces to a linear normed space‖ if and only if there exists in T an open set which is both convex¶ and bounded. In this note, the characterization of other types of spaces among the class of linear topological spaces is studied. Spaces which are *locally bounded*, that is spaces containing a bounded open set, are found to be "pseudo-normed" on the one hand, and metrizable on the other, but not in general normed. Fréchet spaces, or spaces of type (F), are characterized. The main result of the paper is that a linear topological space T is finite dimensional, and hence linearly homeomorphic to a finite dimensional euclidean space, if and only if T contains a compact, open set. This of course is a generalization to linear topological spaces of the well known theorem of F. Riesz for the space of continuous functions.

We first give some needed properties of bounded sets. The following notations will be used throughout. We denote by αS the set of all αx with $x\epsilon S$; by $x+S$, the set of all elements $x+y$ where y ranges over S; by S_1+S_2, the set of all $x+y$ with $x\epsilon S_1$, $y\epsilon S_2$.

THEOREM 1. *A set S of a linear topological space is bounded if and only if, given any neighborhood U of the origin, there is an integer ν such that $|\alpha|<1/\nu$ implies $\alpha S\subset U$.*

* Presented to the Society, April 3, 1937.

† Cf., for example, [1], p. 26. Numbers in brackets refer to the bibliography at the end of the paper.

‡ Cf. [2], pp. 228–229, axioms (A), (B), (C), (5); or [3], pp. 43 and 67.

§ Important instances of a linear topological space were studied by J. von Neumann several years before Kolmogoroff's paper was published. Cf. Mathematische Annalen, vol. 102 (1930), pp. 370–427. See also [5].

‖ Cf. [1], p. 53.

¶ A set S is convex if and only if $x,y\epsilon S$ and $0<\alpha<1$ imply $\alpha x+(1-\alpha)y\epsilon S$.

PROOF. The condition of the theorem is sufficient, for suppose that it is satisfied. Then if $\alpha_\lambda \to 0$, $x_\lambda \epsilon S$, we see, for any chosen neighborhood U of the origin θ, that $\alpha_\lambda x_\lambda \epsilon U$ for all λ for which $|\alpha_\lambda| < 1/\nu$. That is, $\lim_{\lambda \to \infty} \alpha_\lambda x_\lambda = \theta$. The condition is necessary. For, assume that S is bounded, but that the condition of the theorem does not hold. Then for some U and any integer $\nu > 0$, there exists $x_\nu \epsilon S$, and a real α_ν with $|\alpha_\nu| < 1/\nu$ such that $\alpha_\nu x_\nu \bar{\epsilon} U$. Therefore $\alpha_\nu \to 0$ with $1/\nu$, but $\alpha_\nu x_\nu$ does not converge to θ. That is, S is not bounded. This contradiction proves the necessity of the condition.

This theorem provides us with an alternative definition of boundedness. For other equivalent definitions see [5] and [6]. See also [7].

THEOREM 2. *A compact set of a linear topological space is bounded.*

PROOF. Let $S \subset T$ be compact, so that every infinite subset of S has a limit point in T. Assume, contrary to the theorem, that S is not bounded. Let U be any chosen neighborhood of the origin θ. By denying the definition of boundedness, we are led to a sequence $\alpha_\nu \to 0$ and a sequence $x_\nu \epsilon S$, with $\alpha_\nu x_\nu \bar{\epsilon} U$, for $\nu = 1, 2, \cdots$. But, since S is compact, there is a limit point p of the infinite sequence x_ν. By the continuity of the function αx at $\alpha = 0$, $x = p$, there exists a $\delta > 0$ and a neighborhood V of p such that $\alpha V \subset U$ for $|\alpha| < \delta$. Since p is a limit point of the sequence x_ν, there is an infinite subsequence x_{ν_κ} such that $x_{\nu_\kappa} \epsilon V$ for all κ. For sufficiently large κ we have $|\alpha_{\nu_\kappa}| < \delta$, and therefore $\alpha_{\nu_\kappa} x_{\nu_\kappa} \epsilon \alpha_{\nu_\kappa} V \subset U$, contrary to the fact that $\alpha_\nu x_\nu \bar{\epsilon} U$. This contradiction proves the theorem.

DEFINITION. A linear space E will be said to be *pseudo-normed* if corresponding to each $x \epsilon E$ there is a real number $|x|$ with the properties:

(i) $|x| \geqq 0$; $|x| = 0$ implies $x = \theta$,

(ii) $|\alpha x| = |\alpha|\,|x|$ (where $|\alpha|$ is the absolute value of α),

(iii) if $|x| \to 0$, $|y| \to 0$, then $|x+y| \to 0$.

A linear topological space T will be called *pseudo-normable* if it can be pseudo-normed in such a way that the topology according to the pseudo-norm is equivalent to the original Hausdorff topology.

THEOREM 3. *Every pseudo-normed linear space is a locally bounded linear topological space. Conversely, if a linear topological space contains a bounded open set, then the space is pseudo-normable.*

PROOF: Let E be a pseudo-normed space and for any chosen α consider the "sphere" $|y - x| < \alpha$. Denote by $U(x; \alpha)$ the set of points z of this sphere for each of which there exists a positive δ such that

$|w-z|<\delta$ implies $|w-x|<\alpha$. It is easy to verify that E is a Hausdorff space with neighborhoods $U(x;\alpha)$, and that the operations of vector addition and scalar multiplication are continuous, so that E is a linear topological space. Property (iii) of the pseudo-norm is used in proving both the Hausdorff separation axiom and the continuity of addition. Local boundedness now follows from property (ii).

On the other hand, let T be a locally bounded linear topological space, and suppose that U is a bounded open set. Without loss of generality we may suppose that U contains the origin, and that $U=(-1)U$. Then the sets αU, $\alpha\neq 0$, are all open, and clearly form a complete neighborhood system of the origin. Put $|x|=$g.l.b. $|\alpha|$, $x\in\alpha U$, and it follows without difficulty that the postulates for a pseudo-normed space are satisfied. Next, for any integer $\nu>0$ consider the set $|x|<1/\nu$. By definition of $|x|$, this is the set of all x such that $x\in\alpha U$, $|\alpha|<1/\nu$, and hence* the set $|x|<1/\nu$ is open. Now by Theorem 1 and the fact that U is bounded, for any neighborhood V of the origin there is an integer $\nu=\nu(V)$ such that $|\alpha|<1/\nu$ implies $\alpha U\subset V$. Hence the sphere $|x|<1/\nu$, being the union of the sets αU for which $|\alpha|<1/\nu$, is contained in V, and the spheres $|x|<1/\nu$ form a complete neighborhood system of the origin. Thus the topology according to the pseudo-norm is equivalent to the original Hausdorff topology.

COROLLARY 1. *A locally bounded linear topological space satisfies the first countability axiom.*

COROLLARY 2. *If T is locally bounded, then a set $S\subset T$ is bounded if and only if there is a $\mu>0$ such that $x\in S$ implies $|x|<\mu$.*

THEOREM 4. *A necessary and sufficient condition that a linear topological space be finite dimensional is that it contain a non-empty set which is both open and compact.*

PROOF. The necessity is immediate, since every finite dimensional linear topological space (a linear space with a "finite basis" $(x_1, \cdots, x_\nu)$ such that every element is uniquely expressible in the form $\sum_{\lambda=1}^{\nu}\alpha_\lambda x_\lambda$) is linearly homeomorphic to a finite dimensional euclidean space.† To prove the sufficiency, let U be a compact open set containing the origin. By Theorem 2, U is bounded, and hence T is pseudo-normable by Theorem 3. If S is any bounded set of T,

* [7], Theorem 1.2. Special use is made of the continuity of αx in proving this theorem.

† For the proof see [8].

then for some α we have $S \subset \alpha U$ (Theorem 1), so that S is compact, being a subset of the compact set αU. Also, from Corollary 1, every limit point is the limit of a convergent sequence. Hence every bounded sequence has a convergent subsequence, where the convergence is taken according to the pseudo-norm. From here on, with the aid of Corollary 2, the proof follows that of a similar result for the linear normed space of continuous functions due to F. Riesz,* and will be omitted.

COROLLARY 3. *A linear topological space T is linearly homeomorphic to a finite dimensional euclidean space if and only if T is locally compact.*

A theorem on metrizability of "Hausdorff groups" has been proved by G. Birkhoff (see [9]), which states that a Hausdorff group is metrizable if and only if the first countability axiom is satisfied. Since a linear topological space is a commutative Hausdorff group under addition, we are led to the following theorem, characterizing spaces of type (F). For the definition of these spaces see [1], page 35.

THEOREM 5. *A space E is a space of type (F) if and only if E is a sequentially complete† linear topological space which satisfies the first countability axiom.*

PROOF. Let T be a "sequentially complete" linear topological space, satisfying the first countability axiom. Then by G. Birkhoff's metrization theorem for Hausdorff groups, T is metrizable. Moreover the metric (x, y) exhibited by Birkhoff can be easily shown to have the property $(x-y, \theta)=(x, y)$. Therefore T is a linear space with a metric (x, y) satisfying $(x-y, \theta)=(x, y)$ and such that αx is continuous on both the left and the right. Hence, since T is supposed complete, it is a space of type (F).

Conversely, let E be a space of type (F), so that E is a complete metric space with a metric (x, y) which is invariant under translation, and such that the function αx is continuous in α and x separately. In order to prove that E is a linear topological space, it is sufficient to show that αx is a continuous function of its two variables jointly. This may be established by an argument involving sets of the second category, very much like that used by D. Montgomery‡ in a similar connection, and the details will be omitted here.

From Corollary 1 we have the following corollary:

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* [11], p. 75, Lemma 2, and p. 78, Lemma 5.

† [5], p. 10.

‡ [10]. See especially the proof of the lemma on p. 880.

COROLLARY 4. *A locally bounded linear topological space is metrizable by means of a metric satisfying* $(x, y) = (x-y, \theta)$.

From Theorem 3 and Corollary 4 we have the result stated in the introduction, namely, that a locally bounded linear topological space is both pseudo-normable and metrizable, the topologies according to the pseudo-norm, the metric, and the original Hausdorff neighborhoods being equivalent. However, not every locally bounded space is normable, as is shown by the pseudo-normed space $H_{1/2}$ of all infinite dimensional vectors $x = (x_1, x_2, x_3, \cdots)$ for which the series $\sum_{\nu=1}^{\infty} |x_\nu|^{1/2}$ converges, and where $|x| = \{\sum_{\nu=1}^{\infty} |x_\nu|^{1/2}\}^2$. For Tychonoff has shown* that this space is *not* locally convex, and hence not normable, by Kolmogoroff's normability theorem.

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* [8], pp. 768–769.

NOTE ON DEDUCED PROBABILITY DISTRIBUTIONS

R. VON MISES

In this Bulletin, December, 1936, A. H. Copeland* resumed the study of the problem first suggested by H. Poincaré: How can the fact of uniform probability distribution, which we meet so frequently in different games of chance, be explained? Recently E. Hopf devoted a profound essay† to this question and he has just published a short note‡ dealing with his principal results. I want to contribute a quite simple remark which seems to show how far the results are independent of the particular form of dynamical equations.

We assume that there exists a density function $f(x)$ for the one-dimensional variable x, such that $\int_a^b f(x)dx$ denotes the probability that the value of x falls in the interval (a, b) and $\int_{-\infty}^{\infty} f(x)dx = 1$. If between x and y there is established a one-to-one correspondence

$$y = y(x), \qquad x = x(y), \tag{1}$$

the given density function $f(x)$ leads to a new density function $g(y)$ defined by

$$g(y) = f(x)\frac{dx}{dy}. \tag{2}$$

The integral $\int_a^b g(y)dy$ gives, of course, the probability that y belongs to the interval (a, b) and $\int_{-\infty}^{\infty} g(y)dy = 1$.

Now we suppose y to be an "angular" variable, that is, instead of y we consider the new variable:

$$\eta = y - [y], \qquad 0 \leqq \eta < 1, \tag{3}$$

and try to determine the probability distribution $\phi(\eta)$ of η. Evidently, if ν is a positive or negative integer, the probability density of η is given by

$$\phi(\eta) = \sum_{\nu} g(\eta + \nu) = \sum_{\nu} f(x_\nu)\left(\frac{dx}{dy}\right)_{x=x_\nu}; \qquad x_\nu = x(\eta + \nu). \tag{4}$$

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* Vol. 42, p. 895.

† Journal of Mathematics and Physics, Massachusetts Institute of Technology, vol. 13 (1934).

‡ Jahresbericht der Deutschen Mathematiker-Vereinigung, vol. 46 (1936), p. 179.

We may suppose the transformation ratio $dx/dy>0$. In this case the consecutive values $\cdots x(-2), x(-1), x(0), x(1), x(2), \cdots$ define an infinite set of intervals corresponding to intervals of length 1 on the y-axis. Let η' and η'' be two values of η; then the corresponding values $x_\nu'=x(\eta'+\nu)$ and $x_\nu''=x(\eta''+\nu)$ fall in the same interval $(x(\nu), x(\nu+1))$. Therefore, the difference between the values of the products

$$f(x_\nu')\left(\frac{dx}{dy}\right)_{x=x_\nu'} \quad \text{and} \quad f(x_\nu'')\left(\frac{dx}{dy}\right)_{x=x_\nu''}$$

is less than or equal to the variation of the product $f\cdot dx/dy$ through the interval $(x(\nu), x(\nu+1))$, and the difference between the two values of the sum (4) for η' and η'' does not exceed the value of the total variation of the same product. Hence, our theorem follows:

The maximum difference between two values of the deduced probability density $\phi(\eta)$ is less than or equal to the total variation of the product of initial density $f(x)$ by the transformation ratio dx/dy.

If we consider an infinite set of similar problems where the initial distribution $f(x)$ remains unchanged and the transformation ratio is multiplied by a parameter λ, then the deduced distribution $\phi(\eta)$ approaches uniformity as the parameter λ approaches 0 and the functions $f(x)$ and dx/dy are of finite variation.

The mechanical example mentioned by Copeland and by Hopf consists in a system rotating about a vertical axis and subjected to friction forces which depend on the instantaneous angular velocity ω. The dynamical equation is given by

$$\frac{d\omega}{dt} = -r(\omega). \tag{5}$$

Let x be the initial value ω_0 of ω. Until the system comes to rest, a point at the distance 1 from the axis will travel a distance which may be designed by $2\pi y$. Then it follows from (5) that

$$2\pi y = \int \omega dt = \int_0^x \frac{\omega d\omega}{r(\omega)}, \qquad \frac{dx}{dy} = 2\pi\,\frac{r(x)}{x}. \tag{6}$$

Copeland supposes the friction $r(x)$ to be proportional to a parameter λ, but the distribution $f(x)$ to be independent of λ. In this case it is clear that, in consequence of (6), $f(x)\cdot dx/dy$ approaches zero with $\lambda\to 0$, and our theorem shows that the asymptotic value of $\phi(\eta)$ is a constant.

On the other hand, Hopf considers the initial distribution to be given in the form $f(x)=f_1(x-\lambda)$, where f_1 is a function of one variable and λ a parameter. Moreover, he supposes that

$$\lim_{x\to\infty}\frac{r(x)}{x}=0. \tag{7}$$

We find $f(x)\cdot dx/dy=2\pi f_1(x-\lambda)\cdot r(x)/x=2\pi f_1(z)r(\lambda+z)/(\lambda+z)$, which approaches zero, according to (7), as λ increases. If the functions f_1 and r/x are of finite variation, it follows from our theorem that $\phi(\eta)$ approaches uniformity.

It is a quite different question to decide whether the foregoing investigation is or is not sufficient to explain the fact of the nearly perfect uniformity of distribution in a particular case of a real game. Let us consider a sort of roulette consisting of a billiard ball which runs in a smooth circular channel subjected to a constant resistance $r(\omega)=c$; the number of revolutions is found to vary from about 8.1 to 12.1. Our equation (6) gives

$$2\pi y=\frac{x^2}{2c},\qquad \frac{dx}{dy}=\frac{2\pi c}{x}=\left(\frac{\pi c}{y}\right)^{1/2},\qquad x=2(\pi cy)^{1/2}.$$

Therefore x varies from $9(0.4\ \pi c)^{1/2}$ to $11(0.4\ \pi c)^{1/2}$, and if we assume $f(x)$ to be constant in this interval of length $2(0.4\ \pi c)^{1/2}$, we find

$$g(y)=\frac{1}{2(0.4\ \pi c)^{1/2}}\left(\frac{\pi c}{y}\right)^{1/2}=\frac{1}{2(0.4\ y)^{1/2}},\qquad 8.1\leqq y\leqq 12.1.$$

The resulting density function $\phi(\eta)$ is a monotonic decreasing function in the interval from $\eta=0.1$ to $\eta=1.1$. If we divide the whole circle in two parts from $\eta=0.1$ to $\eta=0.6$ and from 0.6 to 1.1, it follows that the probability of a rest position in the first of these semicircles is

$$\int_{8.1}^{8.6}g(y)dy+\int_{9.1}^{9.6}\cdots+\int_{10.1}^{10.6}\cdots+\int_{11.1}^{11.6}\cdots=0.506.$$

The excess of 1.2% is doubtless too large for a fair game of chance. It seems that in such cases other circumstances increase the tendency towards uniformity.

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AN APPLICATION OF SCHLÄFLI'S MODULAR EQUATION TO A CONJECTURE OF RAMANUJAN†

D. H. LEHMER

In 1918 Ramanujan‡ made the following conjecture:

If $q=5$, 7, or 11, and if $24n-1$ is divisible by q^a, then the number $p(n)$ of unrestricted partitions of n is divisible by q^a.

Ramanujan himself proved this conjecture to be true in case‡ $q^a=5$, 7, 5^2, and 7^2, and also§ for $q^a=11$ and 11^2. It has since been proved‖ for $q^a=5^3$. Some modification of the conjecture is necessary, however, since, as Chowla¶ was first to notice, it fails for $q^a=7^3$. In fact, since $24\cdot 243-1=5831$ is divisible by 7^3, it would follow from the conjecture that $p(243)$ is also divisible by 7^3. However, Gupta's table** of $p(n)$ gives

$$p(243) = 13397\ 82593\ 44888,$$

a number†† which is not divisible by 7^3. It occurred to the writer that it would be worth while making an investigation of $p(599)$ and $p(721)$ relative to their divisibility by 5^4 and 11^3 respectively.‡‡ To obtain the value of $p(n)$ for these isolated values of n beyond the limits of then existing tables, use was made of the celebrated Hardy-Ramanujan series,§§ which may be written

$$(1) \qquad p(n) = \frac{(12)^{1/2}}{\mu(24n-1)} \sum_{k=1}^{N} A_k^*(n)(\mu - k)e^{\mu/k} + r_n(N),$$

where we have written μ for $\pi(24n-1)^{1/2}/6$. By taking $N=18$ for $n=599$, and $N=21$ for $n=721$, values were obtained for the series in

† Presented to the Society, September 10, 1937.

‡ Proceedings of the London Mathematical Society, vol. 19 (1919), pp. 207–210; *Collected Papers*, pp. 210–213.

§ Mathematische Zeitschrift, vol. 9 (1921), pp. 147–153; *Collected Papers*, pp. 232–238. A proof for 11^2 is in one of his notebooks.

‖ See Bulletin of the Academy of Sciences, U.R.S.S., 1933, no. 6, pp. 763–800.

¶ Journal of the London Mathematical Society, vol. 9 (1934), p. 247.

** Proceedings of the London Mathematical Society, (2), vol. 39 (1935), p. 149.

†† This number has been verified by the present writer.

‡‡ Journal of the London Mathematical Society, vol. 11 (1936), pp. 114–118.

§§ Proceedings of the London Mathematical Society, (2), vol. 17 (1918), pp. 75–115. Ramanujan's *Collected Papers*, pp. 276–309.

(1) which differed from integers by only .00027 and .00041. Moreover, these integers were divisible by 5^4 and 11^3 as predicted by the conjecture. Still it could not be concluded that these integers were actually $p(599)$ and $p(721)$ since nothing was known about $r_n(N)$ at that time beyond the result of Hardy and Ramanujan that, if $\alpha > 0$,

$$r_n(\alpha n^{1/2}) = O(n^{-1/4}).$$

This uncertainty has since been removed in two ways. In the first place Gupta† has extended his table of $p(n)$ to $p(600)$ and finds the writer's value for $p(599)$; he confirms $p(721)$ with respect to the modulus 247. On the other hand, Rademacher‡ has recently proved that if $(\mu-k)e^{\mu/k}$ is replaced by $(\mu-k)e^{\mu/k}+(\mu+k)e^{-\mu/k}$ in (1), we obtain a convergent series for $p(n)$. With this important fact established it is possible to give an estimate for the remainder $r_n(N)$ and thus the application of the Hardy-Ramanujan series for any particular n is put on a firm basis.

The application of (1) for large values of n was now beset with a further difficulty, namely, that of evaluating the coefficients $A_k^*(n)$. These numbers are usually written $A_k(n)/k^{1/2}$, where the $A_k(n)$ are complicated sums of $24k$th roots of unity arising from the theory of elliptic modular functions. The approximate values of $A_k(n)$ for all n and for $k \leqq 20$ have been tabulated§ by the writer. For large n, however, the series (1) demands more accuracy in the first few A^*'s, and of course additional A^*'s. To meet this problem the writer has made a special investigation‖ of $A_k(n)$, and has obtained the unexpected result that $A_k^*(n)$ is merely a power of 2 times a cosine of a rational multiple of π. As a consequence, if k is small, $A_k(n)$ is a root of a well known algebraic equation of low degree with integer coefficients, so that A_k^* may be easily found with the high degree of accuracy demanded of the first few A^*'s, while if k is large, $A_k^*(n)$ may be obtained from any standard table of natural cosines.

Another consequence is of importance to the estimation of the remainder $r_n(N)$. In fact $A_k^*(n)$ turns out to be much smaller than might be expected. A priori one has as a trivial estimate

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† Proceedings of the Indian Academy of Sciences, vol. 4 (1936), pp. 625–629. His extended table is in Proceedings of the London Mathematical Society, vol. 42 (1937), pp. 547–549.

‡ Proceedings of the London Mathematical Society, (2), vol. 43 (1937), pp. 241–254.

§ Journal of the London Mathematical Society, vol. 11 (1936), pp. 117–118. (Erratum. For $A_{20}(n)$ read $A_{20}(n+5)$.)

‖ To appear in the Transactions of this Society.

$$|A_k^*(n)| \leqq k^{1/2}.$$

One may now prove for example that†

$$|A_k^*(n)| < 2k^{1/3}.$$

Using this estimate one may prove that

$$(2) \quad |r_n(N)| < \frac{\pi^2 N^{7/2}}{3^{1/2}\mu^3}\left\{\sinh\frac{\mu}{N}+\frac{\mu^3}{6N^3}+\left(1+\frac{N}{\mu}\right)(\tfrac{1}{7}+\tfrac{1}{3}\mu^{1/3}N^{-5/3})\right\},$$

a result which is by no means the best possible.

In the application of the Hardy-Ramanujan series there remains only the difficulty of computing e^{μ}. Even when μ is only moderately large the use of the series for e^{μ} is too laborious and risky to be recommended. If μ is not too large, it is possible to evaluate e^{μ} by means of tables of very accurate logarithms such as Wolfram's 48 figure natural logarithms or Sharp's 61 figure common logarithms.‡ This may be done by finding, by continued fractions, an approximation E to e^{μ} so that

$$e^{\mu} = E(1 \pm \epsilon)$$

and such that $\log E$ is a simple combination of tabulated logarithms. Thence

$$\log(1 \pm \epsilon) = \mu - \log E = \pm\delta$$

may be found and will be quite small if E is a good enough approximation. It follows that

$$e^{\mu} = Ee^{\pm\delta} = E\left(1 \pm \delta + \frac{\delta^2}{2!} \pm \frac{\delta^3}{3!} + \cdots\right),$$

where the series converges rapidly. This method of computation, which frequently involves the use of a factor table, was a favorite with Gauss.§

An entirely different method of computing e^{μ} is afforded by the

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† Or indeed even better results than $A_k^*(n) = O(n^{\epsilon})$.

‡ These tables are reprinted in several places such as Callet's *Tables Portatives*, or Peter's *Zehnstellige Logarithmen*. For complete information see Henderson, *Bibliotheca Tabularum Mathematicarum*, vol. 1, part A (*Tracts for Computers*, no. 13), Cambridge, 1926, pp. 195, 204.

§ See for instance his *Werke*, III, pp. 426–432. The fact that many typographical errors exist in the several reprints of the above mentioned tables of logarithms makes it desirable to use two different values of E and thus to make two independent calculations.

theory of elliptic modular functions. The fact that these functions are of practical use in evaluating special exponentials of the type $\exp[\pi D^{1/2}]$ appears to have been noticed first by Hermite† and later by Kronecker.‡ This same fact was rediscovered by Ramanujan§ in an early paper. In the present paper we give a modification of this method to the special problem at hand and give the results of applying it to the calculation of $p(n)$ for $n=1224$, 2052, 2474, and 14031.

In this method we use to the fullest advantage the fact that $\mu=\pi(24n-1)^{1/2}/6$ is of the form $\pi m\nu^{1/2}/6$ where m is a power of 5, 7, or 11, in view of the hypothesis of Ramanujan's conjecture.

In the notation of Weber,‖ the class-invariants f and f_1 are defined by

$$(3)\quad f((-D)^{1/2}) = \exp[\pi D^{1/2}/24]\prod_{k=1}^{\infty}(1+\exp[-(2k-1)\pi D^{1/2}]),$$

$$(4)\quad f_1((-D)^{1/2}) = \exp[\pi D^{1/2}/24]\prod_{k=1}^{\infty}(1-\exp[-(2k-1)\pi D^{1/2}]).$$

Each class-invariant is the root of an algebraic equation with integer coefficients solvable in radicals. The feature of this method is the exploitation of the algebraic character of f and f_1. The derivation of the equation for f or f_1 in case D is large and arbitrary is a difficult matter. In case the number of classes of binary quadratic forms of determinant $-D$ is fairly small, however, the equations can be found rather easily, and in many cases the expressions for f and f_1 in radicals are not difficult either. These equations have been given¶ for a large number of values of D. In case D contains a square factor $D=s^2D'$, it is particularly simple to derive the equations for $f((-D)^{1/2})$ or $f_1((-D)^{1/2})$ from the equation for $f((-D')^{1/2})$ or $f_1((-D')^{1/2})$ by means of the so-called Schläfli's modular equation** of order s as

† Comptes Rendus, vol. 48 (1859), pp. 1101–1102; vol. 49 (1860), p. 18.

‡ Monatsberichte, Akademie der Wissenschaften, Berlin, 1863, p. 44.

§ *Modular equations and approximations to* π, Quarterly Journal of Mathematics, vol. 45 (1914), p. 354. *Collected Papers*, pp. 25–26.

‖ *Elliptische Functionen und Algebraische Zahlen*, Braunschweig, 1891.

¶ Weber, loc. cit., pp. 499–504. Ramanujan, Quarterly Journal of Mathematics, vol. 45 (1914), pp. 350–372; *Collected Papers*, pp. 23–39. G. N. Watson, Quarterly Journal of Mathematics, vol. 3 (1932), pp. 81–98 and 189–212; Proceedings of the London Mathematical Society, (2), vol. 40 (1936), pp. 83–142; Acta Arithmetica, vol. 1 (1936), pp. 284–323; Proceedings of the London Mathematical Society, (2), vol. 42 (1937), p. 377.

** Schläfli, Journal für Mathematik, vol. 72 (1870), pp. 360–369; Weber, loc. cit., pp. 267–274; Berry, American Journal of Mathematics, vol. 30 (1908), pp. 156–159; Watson, Journal für Mathematik, vol. 169 (1933), pp. 238–251.

illustrated below. These transformation equations have been worked out for the first dozen prime values of s as well as for $s=9$, 25, and 49. For our purposes the most important cases are $s=5$, 7, 11, 25, and 49.

The general plan is then as follows. Given n such that $24n-1$ contains q^a, select a divisor D of $24n-1$ such that $(24n-1)/D$ is a square $\geqq 1$. Also D should be chosen large enough to make the infinite products (3) and (4) rapidly convergent and small enough so that either $f((-D)^{1/2})$ or $f_1((-D)^{1/2})$ is readily found either from tables or from one or more applications of Schläfli's modular equation of order p or p^2 to equations already tabulated. Having found this equation we solve it approximately to as many significant figures as are required of e^a. Knowing the left member of either (3) or (4), we easily solve for $\exp[\pi D^{1/2}/24]$, an integer power of which is e^a.

We illustrate the procedure with the example of $n=14031$; in this case $24n-1=11^4\cdot 23$. Selecting $D=11^2\cdot 23$, we first find the modular equation for $f(11(-23)^{1/2})$. We start however with $f((-23)^{1/2})$. If we set

$$f((-23)^{1/2}) = 2^{1/2}x, \tag{5}$$

then†

$$x^3 - x - 1 = 0. \tag{6}$$

If now we set

$$f(11(-23)^{1/2}) = 2^{1/2}y, \quad A = \left(\frac{x}{y}\right)^6 + \left(\frac{y}{x}\right)^6, \quad B = 2xy - \frac{1}{xy},$$

then the Schläfli modular equation of the 11th order is‡

$$A - B^5 + B^3 + 2B = 0,$$

or, in terms of x and y,

$$\begin{aligned} y^{12} - 32(xy)^{11} + 88(xy)^9 - 88(xy)^7 + 44(xy)^5 \\ - 11(xy)^3 + xy + x^{12} = 0. \end{aligned} \tag{7}$$

To obtain the equation satisfied by y, we must eliminate x between (6) and (7). The easiest way to do this is to write (7) with $x=x_1, x_2, x_3$, where these are approximate roots of (6), and to multiply these equations together. Although only approximate values of x are used, the exact integral coefficients of the various powers of y are unmistakably recognized. Thus we readily find that $y=f(11(-23)^{1/2})/2^{1/2}$ satisfies

† Weber, loc. cit., p. 500.

‡ Weber, loc. cit., p. 273.

$$
\begin{aligned}
&y^{36} - 704y^{35} - 1024y^{34} - 31712y^{33} + 36608y^{32} + 89496y^{31} \\
&- 65472y^{30} - 225060y^{29} + 18304y^{28} + 343519y^{27} + 32384y^{26} \\
&- 366080y^{25} - 29051y^{24} + 316096y^{23} + 10120y^{22} - 236808y^{21} \\
&- 4356y^{20} + 156816y^{19} + 17622y^{18} - 91080y^{17} - 17578y^{16} \\
&+ 47047y^{15} + 8095y^{14} - 18975y^{13} - 230y^{12} + 5752y^{11} - 1111y^{9} \\
&- 176y^{8} - 143y^{7} + 396y^{6} - 66y^{5} - 55y^{4} + 78y^{3} + 22y^{2} \\
&- y + 1 = 0.
\end{aligned} \tag{8}
$$

A preliminary reconnaissance shows that the largest root is more than 100 times as large as the absolute value of any other root, so the root squaring method of Dandelin and Graeffe is an especially good method to use in finding this largest root. Moreover, it is not with y but rather with y^4, y^8, y^{16}, and y^{32} that we are concerned, as the following argument shows. In fact, since $\mu = \pi 11^2(23)^{1/2}/6$, we obtain from (8), with $D = 11^2 \cdot 23$,

$$
\begin{aligned}
f^4(11(-23)^{1/2}) &= 4y^4 \\
&= e^{\mu/11}(1 + \exp[-11\pi(23)^{1/2}])^4(1 + \exp[-33\pi(23)^{1/2}])^4 \cdots
\end{aligned}
$$

from which

$$
\begin{aligned}
e^{\mu} &= (4y^4)^{11}(1 + \exp[-11\pi(23)^{1/2}])^{-44} \cdots \\
&= 4^{11}y^{44}(1 - 44\exp[-11\pi(23)^{1/2}] + \cdots).
\end{aligned} \tag{9}
$$

Therefore

$$
\exp[-11\pi(23)^{1/2}] = e^{-6\mu/11} = 4^{-6}y^{-24} + \cdots .
$$

Substituting this in (9) we have

$$
e^{\mu} = 4^{11}y^{44} - 44 \cdot 4^5 y^{20} + \cdots
$$

or

$$
e^{\mu} = 4194304y^4y^8y^{32} - 45056y^4y^{16} + \cdots . \tag{10}
$$

Now the sum of the 128th powers of the roots of (8) differs from y^{128} by a negligible amount. Applying the root squaring method 7 times (retaining only such coefficients as are required), we find S_{128}, and by taking successive square roots we obtain the powers of y needed in (10). Thus we find

$$
\begin{aligned}
e^{\mu} = 90\ &55985\ 87059\ 70450\ 84877\ 28959\ 00188\ 79038\ 39298\ 26173 \\
&19183\ 33361\ 28658\ 25306\ 34561\ 94199\ 90989\ 58100\ 30392 \\
&77156\ 94755\ 49177\ 19559\ 23783\ 93967\ 06869\ 92162.896.
\end{aligned}
$$

In conclusion we give the results of applying the Hardy-Ramanujan series for $p(n)$ when $n=1224$, 2052, 2474, and 14031. If we denote by $S_n(N)$ the sum of the first N terms of the series, so that

$$p(n) = S_n(N) + r_n(N),$$

we find that

$$S_{1224}(16) = 1\ 09765\ 80560\ 41816\ 32042\ 44403\ 44269\ 05625.00262,$$

$$S_{2052}(19) = 20\ 25018\ 07589\ 97203\ 96101\ 85629\ 65051\ 51746\ 73735\ 63574.00612,$$

$$S_{2474}(26) = 14\ 86398\ 21708\ 90271\ 21199\ 41978\ 56277\ 70438\ 51222\ 84071\ 75000.00325,$$

$$S_{14031}(62) = 92\ 85303\ 04759\ 09931\ 69434\ 85156\ 67127\ 75089\ 29160\ 56358\ 46500\ 54568\ 28164\ 58081\ 50403\ 46756\ 75123\ 95895\ 59113\ 47418\ 88383\ 22063\ 43272\ 91599\ 91345\ 00745.00016.$$

If we take the greatest integers in these S's, it is found that the resulting numbers are divisible by the proper powers of 5 and 11 to verify Ramanujan's conjecture. Since this conjecture has been proved for $q^\alpha = 11^2$ and 5^3, it follows that these greatest integers are either true values of the partition function or else differ from such values by multiples of 121 or 125. It follows from (2), however, that this second alternative is impossible since the values of $|r_n(N)|$ are in all cases found to be less than unity. We have therefore established the following facts, all of which are in accord with the conjecture of Ramanujan:

$p(1224)$ is divisible by 5^4,
$p(2052)$ is divisible by 11^3,
$p(2474)$ is divisible by 5^5,
$p(14031)$ is divisible by 11^4.

In no case is the number of partitions divisible by a higher power of 5 or 11 than that indicated.

LEHIGH UNIVERSITY

EXTENSIONS OF FUNCTIONALS ON COMPLEX LINEAR SPACES

H. F. BOHNENBLUST AND A. SOBCZYK

The theorem of Hahn-Banach* states that any linear (continuous) functional defined on a linear subspace of a normed linear space can be extended to the whole space without increasing the norm or modulus of the functional. This theorem however deals only with real linear spaces and real-valued functionals. It is unfortunately not entirely possible to remove the restriction to real values and real coefficients. The present paper discusses this question and investigates the extent to which the theorem of Hahn-Banach remains valid in the general case of complex linear spaces. It is of interest to notice that the difficulties which appear are not due to the possibility that a linear space may be of infinite dimension; the difficulties are present essentially even in the finite dimensional case.

To simplify the exposition let us agree to call a functional $f(x)$ a *real linear* functional if, for any real numbers a, b and any two elements x, y of a linear space L, the relation

$$f(ax + by) = af(x) + bf(y)$$

holds. If the space L is complex linear, and if the above relation remains valid even for all pairs of complex numbers a, b, then the functional will be called *complex linear*. On a complex linear space we may of course have a real linear functional which is not complex linear. Let us remark that by a real subspace of a complex linear space shall be meant any subspace which contains all real linear combinations of any finite number of its elements. Such a subspace may also contain some or all complex linear combinations of its elements. A functional defined over a real subspace will be called *complex linear* if the above relation holds for all complex linear combinations which do not lead to elements outside the subspace. We are of course concerned only with the extension of complex linear functionals.

The theorem of Hahn-Banach has this analogue:

THEOREM 1. *Let l be any complex linear subspace of a normed complex linear space L. Let $f(x)$ be any complex linear functional defined on l, having a norm M. Then there always exists a complex linear functional $F(x)$ defined on L, which coincides with $f(x)$ in l, and which has the same norm M on L.*

* See Banach, *Théorie des Opérations Linéaires*, Warsaw, 1932, p. 28 and p. 55.

PROOF. For $x \in l$, let $f(x) = f_1(x) + if_2(x)$, where $f_1(x)$ and $f_2(x)$ are real-valued. It is easily verified that they are real linear functionals in l of norm $\leqq M$, and that they satisfy furthermore the relation $f_2(x) = -f_1(ix)$. Let $F_1(x)$ be a real linear extension of $f_1(x)$ according to Banach's theorem. Then the equation

$$F(x) = F_1(x) - iF_1(ix)$$

defines a complex linear functional on L, and $F(x)$ coincides with $f(x)$ in l. To verify that the norm of $F(x)$ is M, we notice that, if $F(x) = re^{i\vartheta}$, then

$$|F(x)| = |e^{-i\vartheta}F(x)| = |F_1(e^{-i\vartheta}x)| \leqq M\|x\|.$$

A similar proof is given in a paper by F. J. Murray,* for the case of a functional defined on a linear manifold in an L_p-space.

The subspace l in Theorem 1 is assumed to be a complex linear subspace; that this assumption is an essential one is shown by the following theorem:

THEOREM 2. *In any complex linear space L of infinite dimension, there always exists a real linear subspace l on which there is a complex linear functional $f(x)$ which cannot be continuously extended to the entire space L.*

PROOF. Let $x_1, y_2, \cdots, y_n, \cdots$ be a denumerable sequence of complex linearly independent elements of L, of norm 1, and put

$$z_n = \frac{-nx_1 + y_n}{\|-nx_1 + y_n\|}, \qquad n = 2, 3, \cdots.$$

Then the z_n's and x_1 are complex linearly independent. We construct next by complete induction a sequence $x_1, x_2, x_3, \cdots$ to satisfy the following conditions:

(1) $$x_n = \exp(i\theta_n)z_n, \qquad (\theta_n \text{ real}),$$

(2) $$\|r_1x_1 + r_2x_2 + r_3x_3 + \cdots + r_nx_n\| \geqq r_1$$

for all real numbers $r_1, \cdots, r_n$.

For $n = 1$, we take simply $\theta_1 = 0$. Suppose now that $x_1, x_2, x_3, \cdots, x_{n-1}$ are already determined, and consider in the complex plane w the region R consisting of all complex numbers w for each of which

$$\|x_1 + r_2x_2 + \cdots + r_{n-1}x_{n-1} + wz_n\| < 1$$

* *Linear transformations in* $\mathfrak{L}_p$, $p > 1$, Transactions of this Society, vol. 39 (1936), p. 84.

for a certain choice of real numbers $r_2, r_3, \cdots, r_{n-1}$. This region R, if it exists, is obviously open and convex and does not contain the origin $w=0$. Hence there will exist a straight line passing through the origin which has no point in common with R. Let $w=\exp(i\theta_n)$ be on this straight line, and put $x_n=\exp(i\theta_n)z_n$. The condition (2) is satisfied, trivially for $r_1=0$, and if $r_1\neq 0$ because of the very construction of θ_n. The subspace l, consisting of all real linear combinations of a finite number of $x_1, x_2, \cdots, x_n, \cdots$ is real linear and satisfies the further property that the element

$$c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

does not belong to l if any one c_i is not real. Thus the functional

$$f(x) = c_1$$

is a (real-valued) complex linear functional in l, and is of norm 1 by the property (2) which determined the sequence $\{x_n\}$. This functional cannot be extended to the space L, for if $F(x)$ denotes a complex linear extension and M its norm, we may consider the element

$$nx_1 + \|-nx_1 + y_n\| \exp(-i\theta_n)x_n = y_n.$$

The norm of this element equals 1, and yet the value of $F(x)$ for it must be n, which implies a contradiction if $n>M$.

For a given subspace l of finite dimension, the extension is always possible, although the norm may have to be multiplied by a constant >1. The construction of the above proof shows that no finite upper bound for this constant exists if the space L is of dimension $\geqq 2$.

PRINCETON UNIVERSITY

ON CONTINUED FRACTIONS REPRESENTING CONSTANTS*

H. S. WALL

1. **Introduction.** Let $\xi: x^{(1)}, x^{(2)}, x^{(3)}, \cdots$ be an infinite sequence of points $x=(x_1, x_2, x_3, \cdots, x_m)$ in a space S, and let $\phi_1(x)$, $\phi_2(x)$, $\phi_3(x), \cdots, \phi_k(x)$ be single-valued real or complex functions over S. Then the functionally periodic continued fraction

$$1+\cfrac{\phi_1(x^{(1)})}{1}+\cfrac{\phi_2(x^{(1)})}{1}+\cdots+\cfrac{\phi_k(x^{(1)})}{1}+\cfrac{\phi_1(x^{(2)})}{1}+\cdots+\cfrac{\phi_k(x^{(2)})}{1}+\cfrac{\phi_1(x^{(3)})}{1}+\cdots$$

is a function $f(\xi)$ of the sequence ξ. By a neighborhood of a sequence $\xi: x^{(1)}, x^{(2)}, x^{(3)}, \cdots$, we shall understand a set N_ξ of sequences subject to the following conditions: (i) ξ is in N_ξ; (ii) if $\eta: y^{(1)}, y^{(2)}, y^{(3)}, \cdots$ is in N_ξ, then $\eta_\nu: y^{(\nu+1)}, y^{(\nu+2)}, y^{(\nu+3)}, \cdots$ and $\zeta_\nu: y^{(1)}, y^{(2)}, y^{(3)}, \cdots, y^{(\nu)}, x^{(\nu+1)}, x^{(\nu+2)}, x^{(\nu+3)}, \cdots$ are in N_ξ for $\nu=1, 2, 3, \cdots$.

Let $A_n(\xi)$ and $B_n(\xi)$ be the numerator and denominator, respectively, of the nth convergent of $f(\xi)$ as computed by means of the usual recursion formulas. Put

$$L(\xi, t) = B_{k-1}(\xi)t^2 + [\phi_k(x^{(1)})B_{k-2}(\xi) - A_{k-1}(\xi)]t - \phi_k(x^{(1)})A_{k-2}(\xi).$$

Then our principal theorem is as follows:

THEOREM 1. *Let there be a sequence $c: c^{(1)}, c^{(2)}, c^{(3)}, \cdots$, and a neighborhood N_c of c, and a number r having the following properties:*

(a) *$f(\xi)$ converges uniformly over N_c,*

(b) *$f(c)=r$,*

(c) *$L(\xi, r)=0$ for every sequence ξ in N_c,*

(d) *$\phi_i(x^{(\nu)})\neq 0$, $(\nu=1, 2, 3, \cdots; i=1, 2, 3, \cdots, k)$, for every sequence $\xi: x^{(1)}, x^{(2)}, x^{(3)}, \cdots$ in N_c.*

When these conditions are fulfilled, $f(\xi)=r$ throughout N_c.

The proof of Theorem 1 is contained in §2; §3 contains a specialization and §4 an application of this theorem. In §5 continued fractions

* Presented to the Society, April 9, 1937.

representing constants are obtained by means of certain transformations.*

2. **Proof of Theorem 1.** Let $\eta: y^{(1)}, y^{(2)}, y^{(3)}, \cdots$ be any sequence in N_c. Then $\eta_\nu: y^{(\nu+1)}, y^{(\nu+2)}, y^{(\nu+3)}, \cdots$ is in N_c, and $f(\eta_\nu)$, $(\nu=0, 1, 2, \cdots; \eta_0=\eta)$, converges by (a); and

$$
\begin{aligned}
f(\eta_\nu) &= \frac{A_{k-1}(\eta_\nu)f(\eta_{\nu+1}) + A_{k-2}(\eta_\nu)\phi_k(y^{(\nu+1)})}{B_{k-1}(\eta_\nu)f(\eta_{\nu+1}) + B_{k-2}(\eta_\nu)\phi_k(y^{(\nu+1)})}, \\
f(\eta_{\nu+1}) &= -\frac{B_{k-2}(\eta_\nu)f(\eta_\nu) - A_{k-2}(\eta_\nu)}{B_{k-1}(\eta_\nu)f(\eta_\nu) - A_{k-1}(\eta_\nu)}\phi_k(y^{(\nu+1)}).
\end{aligned} \tag{1}
$$

The determinant of the matrix

$$
\begin{pmatrix} A_{k-1}(\eta_\nu), & A_{k-2}(\eta_\nu)\phi_k(y^{(\nu+1)}) \\ B_{k-1}(\eta_\nu), & B_{k-2}(\eta_\nu)\phi_k(y^{(\nu+1)}) \end{pmatrix}
$$

is $\pm\phi_1(y^{(\nu+1)})\phi_2(y^{(\nu+1)}) \cdots \phi_k(y^{(\nu+1)})$ and is therefore $\neq 0$ by (d). Hence the denominators in (1) cannot vanish, for otherwise the numerators would also vanish, which is impossible. It then follows from (c) that if $f(\eta_\nu)=r$ for one value of ν, then $f(\eta_\nu)=r$ for all values of $\nu(=0, 1, 2, 3, \cdots)$. In particular, if ζ_ν is the sequence $y^{(1)}, y^{(2)}, y^{(3)}, \cdots, y^{(\nu)}, c^{(\nu+1)}, c^{(\nu+2)}, c^{(\nu+3)}, \cdots$, then $f(\zeta_\nu)=r$, $(\nu=1, 2, 3, \cdots)$.

Now by (a), for every $\epsilon>0$ there exists a K such that if $n>K$, $p=1, 2, 3, \cdots$,

$$
\left| \frac{A_{n+p}(\zeta_\nu)}{B_{n+p}(\zeta_\nu)} - \frac{A_n(\zeta_\nu)}{B_n(\zeta_\nu)} \right| < \epsilon \tag{2}
$$

for $\nu=1, 2, 3, \cdots$. Choose a fixed $n>K$, and then choose ν so large that $A_n(\zeta_\nu)/B_n(\zeta_\nu)=A_n(\eta)/B_n(\eta)$. Then on allowing p to increase to ∞ in (2) we find that

$$
\left| f(\zeta_\nu) - \frac{A_n(\eta)}{B_n(\eta)} \right| \leqq \epsilon \quad \text{or} \quad \left| r - \frac{A_n(\eta)}{B_n(\eta)} \right| \leqq \epsilon
$$

if $n>K$. That is, $f(\eta)=r$. Since η was any sequence in N_c our theorem is proved.

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* Leighton and Wall, *On the transformation and convergence of continued fractions*, American Journal of Mathematics, vol. 58 (1936), pp. 267–281; Wall, *Continued fractions and cross-ratio groups of Cremona transformations*, this Bulletin, vol. 40 (1934), pp. 587–592.

3. **Specialization of Theorem 1.** Let the sequence c be such that $f(c)$ is a periodic continued fraction of period k. Let r, s be the roots of the quadratic equation $L(c, t)=0$. Then* in order for $f(c)$ to converge to the value r the following two conditions are both necessary and sufficient, namely:

(α) $B_{k-1}(c)\neq 0$,

(β) $r=s$ or else

$|B_{k-1}(c)r+\phi_k(c^{(1)})B_{k-2}(c)| > |B_{k-1}(c)s+\phi_k(c^{(1)})B_{k-2}(c)|$ and

$A_\lambda(c)-sB_\lambda(c)\neq 0$, $(\lambda=0, 1, 2, \cdots, k-2)$.

An important and simple sufficient condition† for the uniform convergence of $f(\xi)$ over N_c is that

(γ) $|\phi_i(x^{(\nu)})| \leqq \frac{1}{4}$, $(i=1, 2, 3, \cdots, k;\ \nu=1, 2, 3, \cdots)$, for every sequence ξ: $x^{(1)}, x^{(2)}, x^{(3)}, \cdots$ in N_c.

From these remarks and Theorem 1 we then have this result:

THEOREM 2. *Let there be a sequence c and a neighborhood N_c of c such that (γ) and conditions* (c), (d) *of Theorem* 1 *hold. Then if $f(c)$ is a periodic continued fraction of period k, we have $f(\xi)=r$ throughout N_c.*

4. **Application in the case where $\phi_1, \phi_2, \phi_3, \cdots, \phi_k$ are polynomials.** If $k=1$, then $L(\xi, t)=t^2-t-\phi_1(x^{(1)})$, so that in order for (c) of Theorem 1 to hold ϕ_1 must be a constant, and $f(\xi)$ reduces to an ordinary periodic continued fraction.

Let $k=2$. Then $L(\xi, t)=t^2+[\phi_2(x^{(1)})-\phi_1(x^{(1)})-1]t-\phi_2(x^{(1)})$. We shall suppose that $\phi_\nu(x)=\phi_\nu(x_1, x_2, x_3, \cdots, x_m)$, $(\nu=1, 2)$, are polynomials in the real or complex variables $x_1, x_2, x_3, \cdots, x_m$. Let a, b be the constant terms, and G, H the coefficients of $x_1^u x_2^v \cdots x_m^w$ in ϕ_1 and ϕ_2, respectively. Then (c) of Theorem 1 is equivalent to the relations

$$(b-a)r-b=r(1-r), \quad (H-G)r-H=0, \qquad \text{all } G, H.$$

If $r=0$, then $\phi_2\equiv 0$, while if $r=1$, then $\phi_1\equiv 0$. Suppose $r\neq 0, 1$. Then if either G or H is 0, the other is 0 also, and if $G=H$, their common value is 0. Hence (c) of Theorem 1 takes the form of the following identity:

$$r\phi_1 \equiv (r-1)(\phi_2+r), \qquad r\neq 0, 1. \tag{3}$$

On referring to Theorem 2 we now have this result:

THEOREM 3. *Let $\phi_1(x)$ and $\phi_2(x)$ be polynomials in the real or complex variables $x_1, x_2, x_3, \cdots, x_m$ connected by the identity* (3) *with con-*

* Perron, *Die Lehre von den Kettenbrüchen*, 1st edition, p. 276.

† Perron, loc. cit., p. 262.

stant terms a and b, respectively. Let r, in (3), and s be the roots of the quadratic equation $t^2+(b-a-1)t-b=0$ *such that* $r=s$ *or else* $|r+b|>|s+b|$, $s\neq 1$. *Let a, b be such that* $|a|<\frac{1}{4}$, $|b|<\frac{1}{4}$, $a\neq 0$, $b\neq 0$. *Then there exists a positive constant R such that throughout the circle* $|x_i^{(\nu)}|\leqq R$, $(i=1, 2, \cdots, m; \nu=1, 2, \cdots)$, *we have*

$$(4)\quad 1+\frac{\phi_1(x^{(1)})}{1}+\frac{\phi_2(x^{(1)})}{1}+\frac{\phi_1(x^{(2)})}{1}+\frac{\phi_2(x^{(2)})}{1}+\cdots \equiv r,$$

$$x^{(\nu)}=(x_1^{(\nu)}, x_2^{(\nu)}, \cdots, x_m^{(\nu)}).$$

In applying Theorem 2 we have taken $c^{(\nu)}=(0, 0, 0, \cdots, 0)$ in the sequence c. It is to be observed that, when this is done and Theorem 2 applies, the value of the continued fraction depends upon only the constant terms of the polynomials $\phi_1, \phi_2, \phi_3, \cdots, \phi_k$.

5. **Singular continued fractions.** Let T be a transformation which carries the continued fraction $f=x_0+K(x_i/1)$ into another continued fraction $Tf=x_0'+K(x_i'/1)$ in such a way that when either f or Tf converges the other does also and their values are equal. We shall speak of such a transformation as a *proper* transformation of f. Suppose moreover that for some positive integer n the elements x_i of f are subject to the condition

$$(5)\qquad x_i=x_i', \qquad i=n, n+1, n+2, \cdots$$

This gives the following formal relation:

$$x_0+\frac{x_1}{1}+\cdots+\frac{x_{n-1}}{g_n}=x_0'+\frac{x_1'}{1}+\cdots+\frac{x_{n-1}'}{g_n},$$

from which one may compute the value of the continued fraction

$$g_n=1+\frac{x_n}{1}+\frac{x_{n+1}}{1}+\cdots$$

when the latter converges.

The procedure outlined above will now be carried out for the following proper transformation:*

$$T_2:\quad \begin{aligned} x_0' &= x_0+x_1, & x_1' &= -x_1, & x_2' &= (1+x_3)/x_2;\\ x_{2n+1}' &= x_{2n+1}, & x_{2n+2}' &= (1+x_{2n+1})(1+x_{2n+3})/x_{2n+2}, \end{aligned}$$

$$n=1, 2, 3, \cdots; \ x_n\neq 0, -1 \text{ if } n>0.$$

In this case the relations (5) are satisfied if and only if

* Leighton and Wall, loc. cit., p. 277.

$$(6) \qquad x_{2i+2}^2 = (1 + x_{2i+1})(1 + x_{2i+2}), \qquad i = n, n+1, n+2, \cdots,$$

where if $n=0$ the first of these relations is to be replaced by $x_2^2=(1+x_2)$. When $n=0$ we have the relation

$$x_0 + x_1/g_2 = x_0 + x_1 - x_1/g_2$$

from which to compute g_2. It follows that, if f converges, g_2 must converge and have the value 2; and if g_2 converges to a value different from 0, f must converge and $g_2=2$. Moreover, it is impossible for g_2 to have the value ∞, for that would imply that $f=x_0$ while $Tf=x_0+x_1\neq f$. If we now write out the continued fraction g_2 and make a change in notation, the following theorem results.

THEOREM 4. *If $x_1, x_2, x_3, \cdots$ are arbitrary complex numbers $\neq 0, -1$, then the continued fraction*

$$(7) \qquad 1 + \cfrac{e_1(1+x_1)^{1/2}}{1} + \cfrac{x_1}{1} + \cfrac{e_2[(1+x_1)(1+x_2)]^{1/2}}{1} + \cfrac{x_2}{1} + \cfrac{e_3[(1+x_2)(1+x_3)]^{1/2}}{1} + \cdots, \qquad e_i = \pm 1,$$

has one of the values 0 or 2 whenever it converges, and it cannot diverge to ∞.

It is interesting to observe that if $e_i=+1$, (7) is the formal expansion of 2 into a continued fraction by means of the identity

$$1 = \cfrac{(1+t)^{1/2}}{1 + \cfrac{t}{1+(1+t)^{1/2}}}.$$

As a special case we have the expansion

$$(1+N)^{1/2} = 1 + \cfrac{N}{1} + \cfrac{N+1}{1} + \cfrac{N}{1} + \cfrac{N+1}{1} + \cdots,$$

which is valid if N is a positive integer.

The transformation T_2 is one of an infinite group of transformations discussed by the writer* elsewhere in this Bulletin. If one obtains the singular continued fractions corresponding to the case $m=3$ (in the notation of §3, p. 589, of that article), the following three theorems result.

* Wall, loc. cit.

THEOREM 5. *If the continued fraction*

$$1 - \frac{x_1}{1 -} \frac{(x_1^2 - x_1 + 1)}{1} - \frac{x_1}{1 -} \frac{x_2}{1 -} \frac{(x_2^2 - x_2 + 1)}{1} - \frac{x_2}{1 -} \frac{x_3}{1 -} \cdots, \qquad x_n \neq 0,\ x_n^2 - x_n + 1 \neq 0,$$

converges, its value is $(1 \pm i3^{1/2})/2$.

THEOREM 6. *If the continued fraction*

$$1 - \frac{e_1}{1 -} \frac{x_1}{1 -} \frac{(2 - x_1)}{1} - \frac{e_2}{1 -} \frac{x_2}{1 -} \frac{(2 - x_2)}{1} - \frac{e_3}{1 -} \cdots, \qquad e_n = \pm 1,\ x_n \neq 0,\ 2,$$

converges, its value is 0 *or* 1.

THEOREM 7. *If the continued fraction*

$$1 - \frac{x_1}{1 -} \frac{(1 - 2x_1)}{1} - \frac{x_1}{1 -} \frac{x_2}{1 -} \frac{(1 - 2x_2)}{1} - \frac{x_2}{1 -} \cdots, \qquad x_n \neq 0,\ \tfrac{1}{2},$$

converges, its value is 0 *or* $\frac{1}{2}$.

The proofs of these theorems are along the lines of the proof of Theorem 4, and will be omitted.

NORTHWESTERN UNIVERSITY

EXISTENCE THEOREMS FOR SOLUTIONS OF DIFFERENTIAL EQUATIONS OF NON-INTEGRAL ORDER*

EVERETT PITCHER AND W. E. SEWELL

1. **Introduction.** In this paper we prove theorems on the existence and uniqueness of solutions of the differential equation

$$D_x^\alpha y = \phi(x, y), \qquad \alpha > 0, \tag{1.1}$$

where $\phi(x, y)$ is a known function, $y(x)$ is an unknown function, and $D_x^\alpha y$ is the Riemann-Liouville† generalized derivative of order α of the function $y(x)$. For $\alpha = 1$ the equation (1.1) is an ordinary differential equation of the first order and the restrictions on $\phi(x, y)$ for non-integral α are found to be quite similar to those imposed on the function in the integral case.

In establishing the fundamental existence theorem we first prove (§2) a theorem of the kind considered by Birkhoff and Kellogg.‡ Our proof rests on three lemmas which are contained in §3 along with the definition of the generalized derivative. In §4 we establish the existence of a unique solution in the small for $0 < \alpha < 1$. The extension of this solution throughout the region of definition of $\phi(x, y)$ and the case $\alpha > 1$ are considered in §§5 and 6 respectively.

2. **The general existence theorem.** For our purposes the following theorem is fundamental:

THEOREM 2.1. *Let E be a set of continuous functions defined on a common closed interval, and such that if a sequence of functions each belonging to E is uniformly convergent, then the limiting function belongs to E also. Let S be an operator such that if y is in E, then Sy is in E,*

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* Presented to the Society, September 10, 1937.

† B. Riemann, *Gesammelte Mathematische Werke und Wissenschaftlicher Nachlass*, Leipzig, 1892, pp. 331–344; J. Liouville, *Sur quelques questions de géométrie et de mécanique, et un nouveau genre de calcul pour résoudre ces questions*, Journal de l'École Polytechnique, (1), vol. 13, no. 21 (1832), pp. 1–69. For further references see W. E. Sewell, *Generalized derivatives and approximation by polynomials*, Transactions of this Society, vol. 41 (1937), pp. 84–123; we refer to this paper as SI. See also W. Fabian, *Expansions by the fractional calculus*, Quarterly Journal of Mathematics, Oxford Series, vol. 7 (1936), pp. 252–255, where other references are given.

‡ G. D. Birkhoff and O. D. Kellogg, *Invariant points in function space*, Transactions of this Society, vol. 23 (1922), pp. 96–115.

and such that there are constants α and B between 0 and 1 for which, with y_1 and y_2 in E, we have

$$|Sy_1 - Sy_2| \leqq B \max |y_1 - y_2|^{\alpha}. \tag{2.1}$$

Then the equation

$$y = Sy \tag{2.2}$$

has a unique solution in E.

Let y_0 denote a function in E. Define y_n inductively by the equation

$$y_n = Sy_{n-1}, \qquad n = 1, 2, \cdots, \tag{2.3}$$

and consider the series

$$y_0 + (y_1 - y_0) + (y_2 - y_1) + \cdots. \tag{2.4}$$

From (2.1) and (2.3) we have

$$|y_n - y_{n-1}| \leqq B \max |y_{n-1} - y_{n-2}|^{\alpha}, \tag{2.5}$$

and this recurrence yields

$$|y_n - y_{n-1}| \leqq B^{1+(n-2)\alpha} K^m, \tag{2.6}$$

where $K = \max |y_1 - y_0|$ and $m = \alpha^{n-1}$. The test ratio of the series whose general term is the right member of (2.6) is $B^{\alpha}K^{\theta}$ where $\theta = \alpha^n - \alpha^{n-1} = \alpha^{n-1}(\alpha - 1)$. The factor B^{α} is less than 1 and θ approaches 0 as n becomes infinite; consequently series (2.4) converges uniformly.

The function y to which (2.4) converges is a solution of (2.2). For

$$|Sy_n - Sy| \leqq B \max |y - y_n|^{\alpha}$$

and hence Sy_n approaches Sy as n becomes infinite and passage to the limit in (2.3) yields (2.2).

If z is a second solution of (2.2) in E, then

$$|y - z| = |Sy - Sz| \leqq B \max |y - z|^{\alpha}. \tag{2.7}$$

Using (2.7) as a recurrence, we have $y \equiv z$.

3. **Definitions and lemmas.** Let $f(x)$ be a real-valued function defined on an interval* $[k, l]$. The derivative of $f(x)$ of order α, denoted by $D_x^{\alpha} f(x)$, is defined by the following equations:†

$$D_x^0 f(x) = f(x); \tag{3.1'}$$

* We use the square brackets to denote a closed interval.

† $[\alpha]$ is the largest integer $\leqq \alpha$.

$$(3.1'') \quad D_z^{\alpha} f(x) = \frac{1}{\Gamma(-\alpha)} \int_k^x (x-z)^{-\alpha-1} f(z)dz, \qquad \alpha < 0;$$

$$(3.1''') \quad D_z^{\alpha} f(x) = \frac{d^p}{dx^p} D_z^{\alpha-p} f(x), \qquad 0 \leqq [\alpha] = p - 1.$$

It is to be understood in (3.1'') and (3.1''') that $D_z^{\alpha} f(x)$ is defined if and only if the various operations in the order indicated are convergent in the usual sense. In (3.1'') the value of the derivative at $x=k$ is 0. When α is a positive integer, $D_z^{\alpha} f(x)$ in (3.1''') is the ordinary αth derivative of $f(x)$. If α is not an integer, we take that branch of $(x-z)^{-\alpha-1}$ which is real and positive for $x-k$ positive.

If $f(x)$ is bounded and Riemann integrable on $[k, k+h]$, then for α positive $D_x^{-\alpha} f(x)$ exists and is continuous. The value of the derivative at $x=k$ is 0.

Our method of proof depends upon the fact that D_z^{α} and $D_z^{-\alpha}$ are inverse operations under suitable restrictions. Thus we need the following three lemmas. In all three we restrict α to lie between 0 and 1.

LEMMA 3.1. *If $f(x)$ is continuous and possesses a continuous derivative $D_z^{\alpha} f(x)$ on an interval $[k, k+h]$, then $f(k)=0$.*

LEMMA 3.2. *If $f(x)$ is continuous and possesses a bounded derivative $g(x) = D_z^{\alpha} f(x)$ on an interval $[k, k+h]$ then $D_z^{-\alpha} g(x)$ exists on $[k, k+h]$ and is equal to $f(x)$.*

LEMMA 3.3. *If $f(x)$ is continuous on an interval $[k, k+h]$, then $D_z^{\alpha} D_z^{-\alpha} f(x)$ exists on $[k, k+h]$ and is equal to $f(x)$.*

We prove Lemma 3.1 as follows. By definition

$$D_z^{\alpha} f(x) = \frac{d}{dx} D_z^{\alpha-1} f(x).$$

Writing the indicated derivative for $x=k$ as the limit of a difference quotient, we have

$$\lim_{z=k} \frac{1}{(x-k)\Gamma(1-\alpha)} \int_k^x (x-z)^{-\alpha} f(z)dz$$

which becomes, under the transformation $u=(z-k)/(x-k)$,

$$\lim_{z=k} \frac{(x-k)^{-\alpha}}{\Gamma(1-\alpha)} \int_0^1 (1-u)^{-\alpha} f[k+u(x-k)]du.$$

Applying the First Mean Value Theorem for the integral of a product and carrying out the resulting indicated integration, we have

$$\lim_{x=k} \frac{(x-k)^{-\alpha}}{\Gamma(1-\alpha)} \frac{f(\xi)}{1-\alpha}, \qquad k < \xi < x.$$

Since $(x-k)^{-\alpha}$ becomes infinite, it follows that $f(\xi)$ approaches 0, and thus Lemma 3.1 is proved.

Lemma 3.2 follows from Theorem 13.2 of SI. It should be noted here that the hypothesis $f(k)=0$ in Theorem 13.2 is satisfied by virtue of Lemma 3.1.

Lemma 3.3 is included in Theorem 13.1 of SI.

As stated in SI the hypotheses and conclusions of Theorems 13.1 and 13.2 concern subsets of $[k, k+h]$ but it is easy to see that the stronger conclusions follow from our broader hypotheses.

4. **The existence theorem in the small for the case $0<\alpha<1$.** Let the function $\phi(x, y)$ be defined and continuous in a region R of the (x, y)-plane. Let $p(x)$ be defined and continuous at any point x which is the projection of a point of R, and let the point $(k, p(k))$ lie in R.

In the following lemma M, A, a, b, and h are positive constants and $0<\alpha<1$. Furthermore we choose a and b such that points (x, y) for which $0 \leqq x-k \leqq a$ and $|y-p(k)| \leqq b$ belong to R, and we take h less than all of the three numbers

$$a, \qquad \left[\frac{\Gamma(1+\alpha)b}{A}\right]^{1/\alpha}, \qquad \left[\frac{\Gamma(1+\alpha)}{M}\right]^{1/\alpha}.$$

We note for future use that

$$\frac{Ah^{\alpha}}{\Gamma(1+\alpha)} \leqq b, \tag{4.1}$$

and that the number B defined by the equation

$$\frac{Mh^{\alpha}}{\Gamma(1+\alpha)} = B \tag{4.2}$$

is between 0 and 1.

We are now in position to state this lemma:

LEMMA 4.1. *If, for all points (x, y) and (x, y') of R,*

$$|\phi(x, y)| \leqq A, \tag{4.3}$$

$$|\phi(x, y) - \phi(x, y')| \leqq M|y - y'|^{\alpha}, \tag{4.4}$$

then the equation

$$y = p(x) + D_x^{-\alpha}\phi(x, y) \tag{4.5}$$

has a unique continuous solution $y(x)$ on the interval $[k, k+h]$. This solution has the property that $y(k)=p(k)$.

We shall prove the lemma by applying Theorem 1.1. To that end we let E denote the set of all continuous functions $y(x)$ defined on the interval $[k, k+h]$ and satisfying the inequality

$$|\, y(x) - p(x)\,| \leqq b. \tag{4.6}$$

We define the operation S on functions y in E by the relation

$$Sy \equiv p(x) + D_x^{-\alpha}\phi(x, y). \tag{4.7}$$

The function Sy is in E. For Sy is a continuous function of x and

$$|\, Sy - p(x)\,| = |\, D_x^{-\alpha}\phi(x, y)\,| \tag{4.8}$$

$$\leqq \frac{1}{\Gamma(\alpha)}\int_k^x (x-z)^{\alpha-1}\,|\, \phi[z, y(z)]\,|\, dz \tag{4.9}$$

$$\leqq \frac{A}{\Gamma(\alpha)}\,\frac{(x-k)^\alpha}{\alpha} \tag{4.10}$$

$$\leqq b. \tag{4.11}$$

In the above appraisal we obtain (4.10) from (4.9) by using (4.3) and carrying out the integration. We then replace $x-k$ by its upper bound h and use inequality (4.1).

For any two functions y_1 and y_2 in E we have

$$|\, Sy_1 - Sy_2\,| \leqq B \max\,|\, y_1 - y_2\,|^\alpha. \tag{4.12}$$

For

$$|\, Sy_1 - Sy_2\,| \leqq \frac{1}{\Gamma(\alpha)}\int_k^x |\, x-z\,|^{\alpha-1}\,|\, \phi[z, y_1(z)] - \phi[z, y_2(z)]\,|\, dz \tag{4.13}$$

$$\leqq \frac{M \max\,|\, y_1 - y_2\,|^\alpha}{\Gamma(\alpha)}\,\frac{(x-k)^\alpha}{\alpha} \tag{4.14}$$

$$\leqq B \max\,|\, y_1 - y_2\,|^\alpha. \tag{4.15}$$

We obtain (4.14) by using (4.4) and carrying out the integration. Then we replace $x-k$ by its upper bound h and substitute from (4.2).

The hypotheses of Theorem 2.1 are satisfied and the conclusion is that equation (4.5) has a unique solution $y(x)$ in E. It is obvious that $y(k)=p(k)$. We observe that any continuous solution of (4.5) on $[k, k+h]$ is in E. For if y is a solution its value is given by Sy in (4.7) and the argument that Sy is in E still holds.

Using the notation of the lemma and setting $p(x) \equiv 0$ we have the following theorem:

THEOREM 4.2. *If $\phi(x, y)$ satisfies* (4.3) *and* (4.4) *and $0 < \alpha < 1$, then the equation*

$$D_x^{\alpha} y = \phi(x, y) \tag{4.16}$$

has a unique solution $y = u(x)$ on the interval $[k, k+h]$. Furthermore $u(k) = 0$.

From Lemmas 3.2 and 3.3 it is clear that the set of solutions of equation (4.16) and the set of solutions of the equation

$$y = D_x^{-\alpha} \phi(x, y) \tag{4.17}$$

on the interval $[k, k+h]$ are identical. Thus we have Theorem 4.2 by applying Lemma 4.1.

5. **The existence theorem in the large.** The solution $u(x)$ of equation (4.16) can be extended to the boundary of the region R. This is stated precisely in the following theorem:

THEOREM 5.1. *There is a number ζ (possibly $+\infty$) with the following properties:*

(1) *The solution $u(x)$ of equation* (4.16) *whose existence is affirmed in Theorem* 4.2 *can be extended so as to be defined on the interval $k \leqq x < \zeta$.*

(2) *No limit as x approaches ζ of points $(x, u(x))$ belongs to R.*

The method of proof of this theorem is similar to a method sometimes used in proving the corresponding extension to the existence theorem for ordinary differential equations. In brief, let S_ϵ be the region consisting of points of R to the left of the line $x = 1/\epsilon$ whose distance from points of the boundary of R exceeds ϵ. We suppose that ϵ is positive and so small that $(k, 0)$ belongs to S_ϵ. We observe that for any point (x, y) of S_ϵ the numbers a, b of §4 may both be taken as $\epsilon/2$ and consequently a choice h^* of h can be made uniformly for points of S_ϵ.

We need the following lemma.

LEMMA 5.2. *If $u(x)$ is the unique solution of*

$$D_x^{\alpha} y = \phi(x, y) \tag{5.1}$$

on an interval $[k, l]$ with all points $(x, u(x))$ lying in S_ϵ, then $u(x)$ can be extended uniquely as a solution of (5.1) *on the interval $[k, l+h^*]$.*

The equation

$$y = D_x^{-\alpha}\phi(x, y) \tag{5.2}$$

is an equivalent form of (5.1). We observe that solving (5.2) on $[k, l+h^*]$ is equivalent to solving the following pair of equations:

$$y = \frac{1}{\Gamma(\alpha)}\int_k^x (x-z)^{\alpha-1}\phi[z, y(z)]dz, \qquad k \leqq x \leqq l, \tag{5.3'}$$

$$\begin{aligned} y = {} & \frac{1}{\Gamma(\alpha)}\int_k^l (x-z)^{\alpha-1}\phi[z, y(z)]dz \\ & + \frac{1}{\Gamma(\alpha)}\int_l^x (x-z)^{\alpha-1}\phi[z, y(z)]dz, \qquad l \leqq x \leqq l + h^*. \end{aligned} \tag{5.3''}$$

Equation (5.3′) has the unique solution $u(x)$. We write $u(z)$ for $y(z)$ in the first integral of (5.3″), denoting the resulting function by $p(x)$. Then (5.3″) reduces to

$$y = p(x) + {}_lD_x^{-\alpha}\phi(x, y), \qquad l \leqq x \leqq l + h^*, \tag{5.4}$$

where the subscript l indicates that the lower limit of integration in the derivative is l. Equation (5.4) has a unique solution by virtue of Lemma 4.1 and the proof is complete.

By Lemma 5.2 the solution of (5.1) whose existence is established in Theorem 3.1 can be extended to the boundary of S_ϵ. Since ϵ can be chosen arbitrarily small, Theorem 5.1 follows.

6. **The differential equation of order greater than 1.** The equation

$$D_x^\beta y = \phi(x, y), \tag{6.1}$$

where $\beta > 1$, can be handled similarly. We consider this case very briefly. Suppose $p = 1 + [\beta]$. We set $\alpha = \beta - p + 1$. Then equation (6.1) can be written

$$\frac{d^{p-1}}{dx^{p-1}} D_x^\alpha y = \phi(x, y). \tag{6.2}$$

Any solution $y(x)$ of (6.2) is a solution of

$$\begin{aligned} D_x^\alpha y = {} & a_0 + \frac{a_1}{1!}x + \cdots + \frac{a_{p-2}}{(p-2)!}x^{p-2} \\ & + \int_k^x dz_1 \int_k^{z_1} dz_2 \cdots \int_k^{z_{p-2}} dz_{p-1}\phi[z_{p-1}, y(z_{p-1})], \end{aligned} \tag{6.3}$$

for constants a_i properly chosen. Conversely, for any choice of constants a_i any solution of (6.3) is a solution of (6.2). If (k, a_0) is in R and (4.3) and (4.4) are satisfied, it can be shown by applying Theorem 2.1 that equation (6.3) has a unique solution on some interval $[k, k+h]$ and that

$$y^{\alpha+i}(k) = a_i, \qquad i = 0, \cdots, p-2, \tag{6.4}$$

where $y^{\alpha+i}$ is the derivative of order $\alpha+i$. This leads to the following theorem:

THEOREM 6.1. *If β, α, p are numbers as described above, if $\phi(x, y)$ satisfies* (4.3) *and* (4.4), *and if $a_0, a_1, \cdots, a_{p-2}$ is any set of numbers with (k, a_0) in R, then the equation*

$$D_x^\beta y = \phi(x, y) \tag{6.5}$$

has a unique solution satisfying the initial conditions (6.4).

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NOTE ON INTEGRABILITY CONDITIONS OF IMPLICIT DIFFERENTIAL EQUATIONS*

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The Riquier† theory for computing the integrability conditions of a system of partial differential equations of arbitrary order but in a special form gives a precise method for calculating these conditions without repetitions and for obtaining the initial determinations of the solutions. These general arguments imply a corresponding theorem for implicit systems of equations. It is the purpose of the present note to state that theorem and to point out that it is a consequence of the general theory. All references will be to the Janet exposition.

Let F^k, $(k=1, 2, \cdots, m)$, represent a system of differential equa-

* Presented to the Society, December 28, 1934.

† C. Riquier, *Les Systèmes d'Équations aux Dérivées Partielles*, Paris, 1910. M. Janet, *Les systèmes d'équations aux dérivées partielles*, Journal de Mathématiques, (8), vol. 3 (1920), pp. 65–151. J. M. Thomas, *Riquier's existence theorems*, Annals of Mathematics, vol. 30 (1929), pp. 285–310. J. F. Ritt, American Mathematical Society Colloquium Publications, vol. 14, chap. 9.

tions of order k in the n independent variables x and the r dependent variables z. The chain of sets of equations

$$F \equiv F^0, F^1, F^2, \cdots, F^m \tag{1}$$

will be said to be complete if it has the following properties:

(a) The set F^0 contains and implies no equations independent of a z.

(b) The derivatives of the set F^k are among the set F^{k+1}, $(k=0, 1, 2, \cdots, m)$.

(c) Any equations deducible from F^k (integrability conditions) that are of order s will be among the sets F^s, $(s=0, 1, \cdots, k; k=1, 2, \cdots, m)$.

(d) The equations of the set F^{m+1} contain only derivatives of F^m and no equations of type (c) are obtainable from F^{m+1} except the equations already appearing in the sets F^s, $(s=0, 1, \cdots, m)$.

We shall prove the following theorem:

An integrable system is reducible to a complete system (1). *A complete system is integrable.*

The second part of the theorem will be proved first. Each step in the general theory may be carried out for the implicit system. For the purpose of the proof of the general existence theorem and for the determination of initial conditions it is necessary to classify the derivatives appearing in the equations. By the implicit function theorem the higher derivatives may be solved for, and by the method of Janet (chap. 2, §12) the equations may be transformed to satisfy the Janet conditions (a) and (b) (chap. 2, §6). These derivatives appearing in the left members are for convenience associated with a set M of monomials which are the basis for a module determining all principal derivatives.

If m_1 is any monomial of the module M, the product of m_1 by any non-multiplier will be a monomial m_2 belonging to the module. If d_1 is the derivative corresponding to m_1 which is the left member of an equation e_1, then e_2 will be obtainable from (1) by property (b). Consequently the module M may be assumed to be complete. Hence all integrability conditions are in the set in consequence of property (c). The system (1) is, accordingly, integrable. The conditions under which the derivatives corresponding to the modulus M may be solved for by the use of the implicit function theorem constitute additional restrictions under which the general existence theorem is applicable. Under these restrictions the initial conditions may be determined, from the knowledge of the monomials M.

We have shown that (1) is integrable. It remains to demonstrate the possibility of the reduction of an integrable system S to a complete system.

For this purpose we classify the derivatives in the original system S, as we have above, into the classes principal and parametric. The Riquier theory gives us the precise number of integrability conditions, which will be equal to the total number of independent equations obtainable from the derivatives of the set $S \equiv S^0, S^1, \cdots, S^i$. Let the equations obtained by differentiating these or any equivalent equations be designated by the sets $f^0, f^1, \cdots, f^{i+1}$. The sets of equations f^r will be adjoined to the original equations of order r to form a class E^r, $(r=0, 1, \cdots, i+1)$. In dealing with symmetrical sets of equations f^r may be difficult to obtain or to deal with when obtained. It will be satisfactory to include in E^r all results of differentiating S^{r-1} once, together with all rth order equations which are needed to insure the algebraic consistency in the principal derivatives of the derivatives of S^k, $(k=r-1, \cdots, i)$.

This new set E may be treated like the set S. This process must end in any case—in case of integrability or non-integrability. This is apparent from the observation that at each stage new monomials are added to a set which is a basis for a module and the fact that every module has a finite basis.

Accordingly we have the following theorem:

(a) *If the set F^0 is vacuous, the equations* (1) *have a solution with the initial determination given by the Riquier theory.*

(b) *If the set F^0 contains any equations independent of all of the dependent variables, the equations are inconsistent.*

(c) *If the set F^0 contains only equations determining dependent variables z, these functions together with the solutions of the remaining independent equations and the initial determinations constitute the solutions.*

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DIFFERENTIAL INVARIANT THEORY OF ALTERNATING TENSORS*

CLYDE M. CRAMLET

1. **Introduction.** In a former paper† a general method was developed for obtaining a complete system of tensors for a general n-ary q-ic differential form. The quantities $\Lambda_{r_1 \cdots r_{q^s}}$ of that article are proportional to the quantities $a_{r_1 \cdots r_{q^s}}$ of this paper which do not contain the second derivatives when the fundamental tensor is alternating. Thus that method for establishing convariant differentiation with respect to a covariant q-ic form fails when the form is alternating. This exceptional case will be treated here.

Under an analytic transformation of coordinates

$$x^i = x^i(\bar{x}), \qquad i = 1, \cdots, n, \qquad \left|\frac{\partial x^r}{\partial \bar{x}^s}\right| \neq 0, \tag{1}$$

the alternating covariant tensor $a_{r_1 \cdots r_q}$ transforms by the equations

$$\bar{a}_{r_1 \cdots r_q} = a_{\rho_1 \cdots \rho_q} p_{r_1}^{\rho_1} \cdots p_{r_q}^{\rho_q}, \qquad p_s^r = \frac{\partial x^r}{\partial \bar{x}^s}. \tag{2}$$

The property of being alternating is invariant.

We propose to find conditions under which these equations with preassigned $\bar{a}_{r_1 \cdots r_q}$ and $a_{r_1 \cdots r_q}$ admit solutions p_s^r, $|p_s^r| \neq 0$,

$$p_{st}^r = p_{ts}^r, \qquad p_{st}^r \equiv \frac{\partial p_s^r}{\partial \bar{x}^t}, \tag{3}$$

and for which the differential equations

$$\frac{\partial x^r}{\partial \bar{x}^s} = p_s^r \tag{4}$$

are integrable and yield solutions (1) determining a transformation of coordinates.

The statement of the conditions under which such systems admit solutions is contained in a note by the writer which precedes the

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† *The invariants of an n-ary q-ic differential form*, Annals of Mathematics, (2), vol. 31 (1930), pp. 134–150.

present paper in this number of this Bulletin.* The method used here is a direct application of this theory, the $\bar{x}$'s and x's here corresponding to the x's and z's of that paper, and equations (2) here corresponding to the system S of I.C. Here we require the additional restrictions $|p_s^r| \neq 0$.

2. **The derived tensors.** To proceed to the derivation of the sets of equations in the chain,† we differentiate the first set (2), obtaining,

$$
\begin{aligned}
&\frac{\partial \bar{a}_{r_1 \cdots r_q}}{\partial \bar{x}^s} = a_{\rho_1 \cdots \rho_q}\left[p^{\rho_1}_{r_1 s} \cdots p^{\rho_q}_{r_q} + \cdots + p^{\rho_1}_{r_1} \cdots p^{\rho_i}_{r_i s} \cdots p^{\rho_q}_{r_q} + \cdots\right] \\
&\qquad + \frac{\partial a_{\rho_1 \cdots \rho_q}}{\partial x^\sigma} p^{\rho_1}_{r_1} \cdots p^{\rho_q}_{r_q} p^\sigma_s, \\
&\frac{\partial \bar{a}_{sr_1 \cdots r_q}}{\partial \bar{x}^{r_1}} = a_{\rho_1 \cdots \rho_q}\left[p^{\rho_1}_{sr_1} \cdots p^{\rho_q}_{r_q} + \cdots + p^{\rho_1}_{s} \cdots p^{\rho_i}_{r_i r_1} \cdots p^{\rho_q}_{r_q} + \cdots\right] \\
&\qquad + \frac{\partial a_{\sigma \rho_2 \cdots \rho_q}}{\partial x^{\rho_1}} p^{\rho_1}_{r_1} \cdots p^{\rho_q}_{r_q} p^\sigma_s, \qquad (5) \\
&\qquad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots, \\
&\frac{\partial \bar{a}_{r_1 \cdots s \cdots r_q}}{\partial \bar{x}^{r_m}} = a_{\rho_1 \cdots \rho_q}\left[p^{\rho_1}_{r_1 r_m} \cdots p^{\rho_m}_{s} \cdots p^{\rho_q}_{r_q} + \cdots\right. \\
&\qquad \left. + p^{\rho_1}_{r_1} \cdots p^{\rho_m}_{s} \cdots p^{\rho_k}_{r_k r_m} \cdots p^{\rho_q}_{r_q} + \cdots\right] + \frac{\partial a_{\rho_1 \cdots \sigma \cdots \rho_q}}{\partial \bar{x}^{\rho_m}} p^{\rho_1}_{r_1} \cdots p^{\sigma}_{s}, \\
&\qquad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots.
\end{aligned}
$$

This last equation is written for $k > m$. A similar term appears for $k < m$.

Consider any term involving $p^\alpha_{r_i s}$. It appears in the first equation in

$$a_{\rho_1 \cdots \rho_q} p^{\rho_1}_{r_1} \cdots p^{\rho_i}_{r_i s} \cdots p^{\rho_q}_{r_q}, \qquad i = 1, \cdots, n.$$

This term appears also in the $(i+1)$th equation as, $a_{\rho_1 \cdots \rho_q} p^{\rho_1}_{r_1} \cdots p^{\rho_i}_{sr_i} \cdots p^{\rho_q}_{r_q}$. All such terms will cancel if the sum of all equations after the first is subtracted from the first.

Now consider any terms involving $p^\alpha_{r_k r_m}$, $r_k \neq r_m \neq s$. There will be two such terms, one appearing in the $(m+1)$st equation and another in the $(k+1)$st. It will be convenient to think of $m < k$ but this is not a restriction. The terms referred to are

* *Note on integrability conditions of implicit differential equations*, this Bulletin, vol. 44 (1938), pp. 107–109. This paper will be referred to as I.C.

† Equation (1), I.C.

$$a_{\rho_1\cdots\rho_q} p_{r_1}^{\rho_1} \cdots p_s^{\rho_m} \cdots p_{r_k r_m}^{\rho_k} \cdots p_{r_q}^{\rho_q}$$

and

$$a_{\rho_1\cdots\rho_q} p_{r_1}^{\rho_1} \cdots p_{r_m r_k}^{\rho_m} \cdots p_s^{\rho_k} \cdots p_{r_q}^{\rho_q}.$$

By an interchange of summation indices ρ_m and ρ_k it appears from the alternating character of $a_{r_1\cdots r_q}$ that these terms differ only in sign, hence cancel in the sum of all equations after the first. Thus

$$\bar{a}_{r_1\cdots r_q s} = a_{\rho_1\cdots\rho_q\sigma} p_{r_1}^{\rho_1} \cdots p_{r_q}^{\rho_q} p_s^{\sigma}, \tag{6}$$

$$a_{r_1\cdots r_q s} \equiv \frac{\partial a_{r_1\cdots r_q}}{\partial x^s} - \frac{\partial a_{sr_2\cdots r_q}}{\partial x^{r_1}} - \cdots - \frac{\partial a_{r_1\cdots r_{q-1}s}}{\partial x^{r_q}}. \tag{7}$$

These equations may be written

$$a_{r_1\cdots r_q s} = \frac{\partial a_{r_1\cdots r_q}}{\partial x^s} \pm \frac{\partial a_{r_2\cdots r_q s}}{\partial x^{r_1}} + \frac{\partial a_{r_3\cdots s r_1}}{\partial x^{r_2}} \pm \cdots, \tag{8}$$

where all signs are positive if q is even, and alternately positive and negative if q is odd, and each term is obtained from the preceding term by a cyclic advance of the indices.

This is the well known* alternating tensor which will be called the first derived tensor. It is also well known that when the tensor appearing in the right member of (8) is a derived tensor the result vanishes identically, or the second derived tensor has the value

$$a_{r_1\cdots r_q s t} \equiv 0. \tag{9}$$

3. **A complete system of eliminants.** Equations (6) are eliminants of (5), consequences of (3). There are no other eliminants, when we assume that (2) and (6) possess simultaneous solutions p_s^r. For (6) may be used to replace the first set of (5). The remaining q sets contain given derivatives $p_{sr_1}^{\rho}$ in but a single set, and the $p_{r_1 s}^{\rho}$ do not appear. Hence no eliminants due to (3) exist. No eliminants independent of (3) are obtainable, for these equations are consistent from the hypothesis that (2) are consistent. Equations (6) must be similarly treated. The resulting equations are like (5) with q replaced by $q+1$ and will be referred to as (5′). Again the first set may be discarded by the use of (9) and the only equations containing the derivatives $p_{sr_1}^{\rho}$ under discussion above will appear in a single set. The algebraic eliminants obtained by eliminating the $p_{sr_1}^{r}$ from (5) and (5′) will be satisfied because of the hypothesis of consistency of (2) and (6) and no eliminants can result from the combined sets (5) and

* Murnaghan, *Vectors and Tensors*, p. 35.

(5′) because of (3), for we have shown that, if we consider any derivatives p^r_{st}, the independent equations of (5) and (5′) may be so chosen that only p^r_{st} and not p^r_{ts} appear in them.

4. **The complete chain.** The chain of I.C. (1) may now be constructed. The set F^1 will include the sets (2) and (6):

$$
\begin{aligned}
F^1: \quad & \bar{a}_{r_1 \cdots r_q} = a_{\rho_1 \cdots \rho_q} p^{\rho_1}_{r_1} \cdots p^{\rho_q}_{r_q}, \\
& \bar{a}_{r_1 \cdots r_q s} = a_{\rho_1 \cdots \rho_q \sigma} p^{\rho_1}_{r_1} \cdots p^{\rho_q}_{r_q} p^{\sigma}_{s}.
\end{aligned} \tag{10}
$$

The set F^2 contains only derivatives of F^1; no new equations implied by (3) are obtainable. Therefore, the chain terminates with F^1 and *if there exist solutions* p_s^r *of* (10), $|p_s^r| \neq 0$, *these solutions may be so chosen that* (4) *are integrable.* The eliminants of the set F^1 are necessary conditions for the solution of the set F^1. These eliminants will be of the form of the equality of absolute algebraic invariants or the vanishing of tensors. All such will comprise the set F^1 and will be finite equations that must be satisfied.

We have reduced the problem to the algebraic form: *The necessary and sufficient condition that two differential alternating tensors* $a_{r_1 \ldots r_q}$ *and* $\bar{a}_{r_1 \ldots r_q}$ *be equivalent is that the pairs of alternating tensors* $a_{r_1 \ldots r_q}$, $a_{r_1 \ldots r_q s}$ *and* $\bar{a}_{r_1 \ldots r_q}$, $\bar{a}_{r_1 \ldots r_q s}$ *be algebraically equivalent.*

A complete set of tensors is a set in terms of which the equivalence problem is expressible. The tensors $a_{r_1 \ldots r_q}$ and $a_{r_1 \ldots r_q s}$, therefore constitute a complete set of tensors of the basic tensor.

5. **On covariant differentiation.** From the equations (7) defining the $a_{r_1 \ldots r_q s}$ in terms of the derivative of the $a_{r_1 \ldots r_q}$, it might be supposed that the relations were reciprocal. That they are not may be seen by a count of the related quantities, the $a_{r_1 \ldots r_q s}$ being antisymmetric in $q+1$ indices while the derivatives of the $a_{r_1 \ldots r_q}$ are antisymmetric in but q indices.

Since the number of independent components of the first derived tensor is less than the number of first partial derivatives of the basic tensor, we may infer that covariant differentiation is impossible. For, if first covariant derivatives were obtainable, the number of independent covariant derivatives would agree with the number of first derivatives and the existence of solutions p_s^r of the equations (10) relating the fundamental system of tensors in two coordinate systems would imply that these same p's would satisfy the transformation equations of (2) and their covariant derivatives. But this is impossible, for the latter set would be composed of a greater number of independent equations.

6. **Significance of derived tensor.** The analytic significance of the vanishing of the first derived tensor may be easily demonstrated. An interpretation is suggested by the observation that constant values of the base tensor imply the vanishing of the first derived tensor.

Let the given tensor be $a_{r_1 \ldots r_q}$ satisfying $a_{r_1 \ldots r_q s} = 0$. We may now regard the equations (2) as equations of condition, the $\bar{a}$'s unassigned and the p_s^r to be determined. Choose arbitrary constant p's such that $|p_s^r| \neq 0$, fixed a's at a point P, and determine the corresponding $\bar{a}$'s. Define these constant values to be the $\bar{a}$'s in the neighborhood of P. The $\bar{a}_{r_1 \ldots r_q s}$ will vanish identically and the second set of equations (10) is identically satisfied. Therefore the chain is completed with the single set (2), which, regarded as differential equations, are integrable. For the constants were chosen so that $|p_s^r| \neq 0$ for constant values of the p's at the arbitrary point P, and from continuity considerations the p's have values in the neighborhood satisfying this condition and equations (2). From the general theorem it follows that these quantities may be selected so that the equations (4) are integrable, yielding (1).

We have proved that *the vanishing of the tensor* $a_{r_1 \ldots r_q s}$ *is a necessary and sufficient condition that a coordinate system exist in which the* $a_{r_1 \ldots r_q}$ *are constants*. From the theorem that the second derived tensor of an alternating tensor vanishes identically we have the following interesting restatement of the present theorem: *A coordinate system always exists in which a derived tensor is constant.*

From the remark that the first derived tensor of a completely alternating tensor (of order n) vanishes identically we infer the corollary of the above theorem: *A coordinate system exists for which a completely alternating tensor is constant.*

7. **Remark on generalized Green's Theorem.** From this theorem the well known extension of Green's Theorem may be inferred.* The problem of stepping down the invariant integral of order $q+1$ to an integral of order q requires a condition which may be expressed in the language of this paper as requiring that the given tensor B determining the integrand of the integral of $(q+1)$th order be a derived tensor† of some tensor A. In this case coordinates may be chosen for which the integrand is constant and the integration is immediate.

UNIVERSITY OF WASHINGTON

* Philip Franklin, *Multiple integrals in n-space*, Annals of Mathematics, (2), vol. 24 (1922–1923), pp. 213–226.

† See equations (37) of Franklin's paper.

NOTE ON CONVEX REGIONS ON THE SPHERE*

R. M. ROBINSON

By a convex region on the sphere we mean a region such that any great circle arc of length less than 180°, whose end points lie in the region, lies entirely in the region. Let G denote any convex region, G_0 the diametrically opposite region, and G_1 the set obtained from the whole sphere by excluding G and G_0 together with their boundaries. Let $\rho = \arctan(3^{1/2}/2)$; then $40°53' < \rho < 40°54'$. We shall prove the following theorem.†

There is on the sphere a circle-interior of radius ρ (measured on the sphere), which lies entirely in G or entirely in G_1. The number ρ cannot be replaced by any larger number.

Let r be the least upper bound of the radii of circle-interiors lying in G. Then for every integer $n > 1$ there is a circle-interior of radius $r(1-1/n)$ lying in G. A limit point of their centers is the center of a circle-interior C of radius r lying in G. We may suppose that G is neither the whole sphere nor a hemisphere, so that $r < 90°$.

No closed semicircumference forming part of the boundary of C can be free of boundary points of G. For if it were, it would be at a distance $d > 0$ from the boundary of G. Thus G would contain not only C but all points within a distance d from one-half of its circumference. Hence G would contain a circle-interior of radius greater than r.

If P is any boundary point of G on the circumference of C, then G lies entirely on one side of the great circle tangent to C at P. For if there were a point P' of G on the opposite side of this great circle from C, we could join P' to P by a great circle arc of less than 180°, which when extended through P would cut C. Taking P'' as a point sufficiently near to P on $P'P$ extended, we should have $P'P'' < 180°$, P on $P'P''$, P' and P'' in G, therefore P in G, contrary to hypothesis.

We shall show that there is an arc PQ forming not more than one-half nor less than one-third of the circumference of C, and whose end points P and Q are boundary points of G. Let P_1 be any boundary point of G on the circumference of C, and let P_0 be the opposite point of the circumference. If P_0 is a boundary point of G, then we may take either arc P_1P_0 as PQ. Otherwise let P_2 and P_3 be the boundary

* Presented to the Society, November 27, 1937.

† This theorem was suggested to me by Hans Lewy, who makes use of it in a current paper.

points of G on the circumference of C nearest to P_0 on either side. Then the arcs P_1P_2 and P_1P_3 are less than a semicircumference by construction, and P_2P_3 cannot be greater than a semicircumference, since its interior is free of boundary points of G. The longest of the three arcs is at least one-third of the circumference, and hence can be taken as PQ.

Drawing the tangent great circles to C at P and Q, we have G included within a lune, whose angle can be easily calculated (in terms of r and the length of PQ). Let 2θ be the angle of the lune, and 2α the angle subtended by PQ at the center of C. Then $60° \leqq \alpha \leqq 90°$. Join the nearer point of intersection of the tangents at P and Q to the center of C; also join P to the center of C. A right triangle is formed with angles α and θ, and side r opposite the latter angle. Applying a formula true for any right spherical triangle, we have

$$\cos\theta = \cos r \sin\alpha,$$

hence $\cos\theta \geqq (3^{1/2}/2)\cos r$. Let $r' = 90° - \theta$. Then in either of the supplementary lunes there is a circle-interior of radius r', and

$$\sin r' \geqq (3^{1/2}/2)\cos r.$$

For values of r and r' satisfying this inequality, the smallest value of $\max(r, r')$ will occur for $\sin r' = (3^{1/2}/2)\cos r$, and $r' = r$. The common value of r and r' is then $\rho = \arctan(3^{1/2}/2)$. Hence in every case $\max(r, r') \geqq \rho$, so that there is always a circle-interior of radius ρ in G or G_1.

We must show that ρ cannot be replaced by any larger number. For this purpose we find a particular region G so that there is no circle-interior of radius greater than ρ in G or G_1. Let G be the interior of an equilateral triangle circumscribed about a circle of radius ρ. If the angles of the triangle are 2θ, we have (putting $r = \rho$ and $\alpha = 60°$) $\cos\theta = (3^{1/2}/2)\cos\rho = \sin\rho$, or $\theta = 90° - \rho$. The angles of the triangle being more than 90°, the sides are also more than 90°. The exterior angles of the triangle are 2ρ. In G_1 there are then six circles of radius ρ, touching one side of G and one side of G_0. There are no larger circles in G_1. For every maximum circle, since it cannot touch G and G_0 at three points, must touch them at two opposite points of the circle. This can happen only for the six circles mentioned and for the six circles which pass through a vertex of one triangle and are tangent at the mid-point of a side of the other. These latter circles are seen, however, not to furnish even a relative maximum.

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ON CERTAIN EQUATIONS IN MATRICES WHOSE ELEMENTS BELONG TO A DIVISION ALGEBRA*

M. H. INGRAHAM

1. **Introduction.** A method was given by the author† to determine all matrices X having elements in a field F satisfying the matric equation $P(X)=A$, where $P(\lambda)$ is a polynomial with coefficients in F and A is a matrix with elements in F. The result gives X to within a similarity transformation commutative with A. The purely formal generalization of allowing F to be a division algebra, possibly non-commutative, not only leads to difficulties that probably can not be handled by extensions of the methods of the above mentioned paper, but seems to be devoid of interest. However, if we consider that A defines a linear transformation, we see that the answer to the following question may be of interest: Given the constants $a_n, a_{n-1}, \cdots, a_0$, for what matrices X is $\sum_0^n X^i \xi a_i = A\xi$ for every vector ξ? If the numbers involved lie in a field, this reduces to the previously discussed problem.

After defining the necessary notation, this paper proceeds to give the solution of a slightly more general problem.

Consider a division algebra D and an $n \times n$ matrix A with elements in D. Let $g(\lambda)=\sum \lambda^i a_i$ be a polynomial in λ with coefficients a_i in D. If ξ is an $n \times 1$ matrix (vector), with elements in D, then $g(A) \odot \xi$ is defined‡ to be $\sum A^i \xi a_i$.

If g_1 and g_2 are the two polynomials $\sum \lambda^i a_{1i}$ and $\sum \lambda^i a_{2i}$, respectively, then $g_1 \odot g_2 = \sum \lambda^i g_2(\lambda) a_{1i}$.

The transformation defined by $\xi_1 = g(A) \odot \xi$ will be right linear if and only if the coefficients of g are in the centrum C of D where C denotes the totality of elements of D commutative with every element of D.

Consider two polynomials P and Q with coefficients in D. Let A be an $n \times n$ matrix with elements in D. It is the purpose of this paper to give methods for finding all solutions X of the equation

* Presented to the Society, September 8, 1937. In the preparation of this paper the author was aided by M. C. Wolf, who acted as his research assistant under appointment authorized by the Research Committee of the University of Wisconsin.

† M. H. Ingraham, *On the rational solutions of the matrix equation* $P(X)=A$, Journal of Mathematics and Physics, vol. 13 (1934), pp. 46–50.

‡ See M. H. Ingraham and M. C. Wolf, *Relative linear sets and similarity of matrices whose elements belong to a division algebra*, Transactions of this Society, vol. 42 (1937), pp. 16–31; referred to herein as *Relative linear sets*.

$$P(X) \odot \xi = Q(A) \odot \xi, \qquad (\xi),$$

where X is an $n \times n$ matrix with elements in D. The (ξ) is used throughout in displayed equations for the phrase "for every ξ." This problem reduces to the problem of factorization of polynomials and another problem described in §4.

2. **Reduction of problem to consideration of linear transformations only.** We prove the following theorem:

THEOREM 1. *If a_i, $(i=1, \cdots, k)$, elements of a division algebra D, are linearly independent as to coefficients in the centrum of D and if M_i, $(i=1, \cdots, k)$, are matrices with elements in D satisfying the relation*

$$\sum M_i \xi a_i = 0, \qquad (\xi),$$

then $M_i=0$ for every i.

This theorem follows at once if it can be established when the ξ is limited to be of the form $\delta_j d$ where δ_j is the Kronecker delta vector and d is in D. That this is true may be established by means of the following lemma:

LEMMA A. *If a_i, $(i=1, \cdots, k)$, elements of a division algebra D, are linearly independent as to coefficients in the centrum of D and if $b_1, b_2, \cdots, b_k$ form a set of elements of D satisfying the relation $\sum b_i d a_i = 0$ for every d in D, then $b_i=0$ for every i.*

The proof is by induction on k.

There is no loss of generality in letting $a_1=1$. Let e be an element of D not commutative with a_2. Both of the following relations are satisfied for every d:

$$\sum b_i d e a_i = 0, \qquad \sum b_i d a_i e = 0,$$

and hence

$$\sum_{i \neq 1} b_i d(e a_i - a_i e) = 0.$$

If the left-hand member of the above equation is expressed as a linear form in a set of the $(ea_i - a_i e)$ which are linearly independent as to coefficients in the centrum, the induction hypothesis shows that the coefficients of this set will all be zero. Hence some linear combination $\sum b_i c_i$ will be zero, where the c_i are in the centrum and $c_1=0$. Moreover, c_2 may be taken to be 1. Since the c_i are in the centrum, it follows that $\sum b_i d c_i a_2 = 0$ for every d, and by subtracting this from $\sum b_i d a_i = 0$ it follows that

$$b_1 d a_1 + \sum_{i>2} b_i d(a_i - c_i a_2) = 0.$$

From the induction hypothesis $b_1 = 0$, and hence by induction $b_i = 0$, $(i = 2, \cdots, k)$.

This lemma is also a consequence of a theorem proved by the author* that a set of finite order which is both right and left linear relative to coefficients in D has a commutative base.

Another form of the above lemma is as follows:

If ξ is a vector whose elements belong to a division algebra D and are linearly independent as to coefficients in the centrum of D, there exists a vector η with elements in D such that the transpose $(\xi\eta')'$ of the dyad $\xi\eta'$ is non-singular.

Consider

$$P(X) \odot \xi = Q(A) \odot \xi, \qquad (\xi),$$

and let $1, \alpha_2, \cdots, \alpha_k$ be a proper base for the division algebra D over its centrum C. This equation may be written in the form

$$\sum P_i(X)\xi\alpha_i = \sum A_i \xi\alpha_i, \qquad (\xi),$$

where the P_i have coefficients in the centrum, and hence, from Theorem 1, X must satisfy the equations

$$P_i(X) = A_i, \qquad (i).$$

3. **The solution of $P(X) = A$ where P has coefficients in the centrum.** Let P be a polynomial with coefficients in the centrum C of the division algebra D. If Y is a matrix such that $P(Y)$ is similar to A, then the transformation that takes $P(Y)$ into A takes Y into a matrix X such that $P(X) = A$. All solutions of $P(X) = A$ similar to X are transforms of X by non-singular matrices commutative with A. The problem of the similarity of two matrices whose elements belong to a division algebra has been discussed by Jacobson† and also by the author‡ in collaboration with M. C. Wolf. The following statements are based upon the results of these papers.

If g is a polynomial in D, then there exists a polynomial h of least degree with coefficients in the centrum C and leading coefficient unity for which g is an interior (left-hand) factor. If g is irreducible in D,

* M. H. Ingraham, *General theory of linear sets*, Transactions of this Society, vol. 27 (1925), pp. 163–196.

† N. Jacobson, *Pseudo-linear transformations*, Annals of Mathematics, (2), vol. 38 (1937), p. 485.

‡ *Relative linear sets.*

then h is irreducible in C. In this case any other polynomial irreducible in D which is an interior factor of h is of the same degree as g. The degree of g, $\langle g \rangle$, is also symbolized by $\langle\langle h \rangle\rangle$, that is, the degree of the minimum polynomial defining h.

The nullity of a matrix M is the order of the vector space orthogonal to M, that is, the nullity is equal to the number of right linearly independent vectors ξ satisfying $M\xi = 0$. It may be remarked that, since the rank plus the nullity of a matrix is equal to its order, a knowledge of the nullity of M is equivalent to a knowledge of the rank of M.

For any matrix M there exists a minimum polynomial h with coefficients in the centrum C for which $h(M) = 0$. Consider $h = \prod h_i^{k_i}$ where the h_i are polynomials over C irreducible in C. If also $h(N) = 0$ and if the nullity of $h_i^t(N)$ is the same as $h_i^t(M)$ for every i and $t \leqq k_i$, then M is similar to N. This condition is also necessary. Moreover, the second difference of the nullity of $h_i^t(M)$ as a function of the exponent t is always zero or negative. Since the nullity of $h_i^0(M)$ is zero and since for sufficiently large t the first and second difference of the nullity remains zero, it is clear that a knowledge of the second difference of the nullity of $h_i^t(M)$ for all i and $t \leqq k_i$ is sufficient to determine the class of similar matrices to which M belongs.

Suppose f is a polynomial over C irreducible in C, and suppose that $f^{k_1}, f^{k_2}, \cdots, f^{k_r}$ are the powers of f in the characteristic divisors of M, that is, the highest powers of f in the various invariant factors of M when the latter are factored into polynomials irreducible in C. Let $n(l)$ be the number of k_i equal to l. For every pair of positive integers s and t the nullity of $f^{st}(M)$ is equal to

$$\langle\langle f \rangle\rangle \left[\sum_{l=1}^{st} ln(l) + st \sum_{l>st} n(l) \right].$$

The second difference of this nullity considered as a function of t is

$$(1) \qquad - \langle\langle f \rangle\rangle \left\{ \sum_{l=1}^{s} [ln(st + l) + (s - l)n(st + s + l)] \right\}.$$

If $s = 1$, this reduces to

$$(1') \qquad - \langle\langle f \rangle\rangle n(t + 1).$$

Returning to the problem of finding a matrix Y such that $P(Y)$ is similar to A, let h be the minimum polynomial of A with coefficients in the centrum, and let $h = \prod h_i^{k_i}$, where the h_i are distinct polynomials with coefficients in the centrum and irreducible in the

centrum. Let the characteristic divisors of A be $h_i^{q_{ik}}$. Moreover, let

$$h_i(P) = \prod f_j^{l_{ij}}$$

and let Y be a matrix with characteristic divisors $f_j^{p_{jk}}$. Let $m_j(s)$ and $n_j(s)$ equal, respectively, the number of the q_{jk} and p_{jk} which are equal to s. If $P(Y)$ is to be similar to A, the second difference of the nullity of $[h_i(P(Y))]^t$ must be the same as the second difference of the nullity of $[h_i(A)]^t$. Using (1) and (1′) and the fact that the nullity of the product of two relatively prime polynomials in a matrix is the sum of the nullities of the two polynomials in the given matrix taken separately, we see that this yields the diophantine equations

$$(2)\quad \sum_j \langle\langle f_j \rangle\rangle \sum_{l=1}^{l_{ij}} [ln_j(tl_{ij} + l) + (l_{ij} - l)n_j(tl_{ij} + l_{ij} + l)] = \langle\langle h_i \rangle\rangle m_i(t+1), \qquad (i, 0 \leqq t \leqq k_i).$$

This system of diophantine equations for the $n_j(s)$ can have only a finite number of positive integral solutions, each of which will define a matrix Y similar to a solution of $P(X) = A$, and any two solutions define dissimilar Y's. Moreover, every solution X will be similar to some Y thus defined.

4. **The equation** $P(X) \odot \xi = Q(A) \odot \xi$. It has been proved that this equation may be reduced to a system of equations $P_i(X) = A_i$, where the coefficients of the P_i are in the centrum. The solution of these equations having been reduced to the factorization of polynomials over a division algebra, there still remains the problem of picking out their simultaneous solutions. Consider the case of two equations. Let $X_{11}, X_{12}, \cdots, X_{1k}$ be a complete set of dissimilar solutions of $P_1(X) = A_1$. Any transform of X_{1i} by a non-singular matrix commutative with A is a solution of this equation, and all solutions are of this form. Let $X_{21}, X_{22}, \cdots, X_{2l}$ be a complete set of dissimilar solutions of $P_2(X) = A_2$. If X_{1i} is dissimilar to all the X_{2j}, then there is no simultaneous solution of the two equations similar to X_{1i}. Each of $P_1(X) = A_1$ and $P_2(X) = A_2$ defines an equation (2), and, of course, only such $n_j(l)$ as are simultaneous solutions for these equations (2) need be used. Suppose X_{11} is similar to X_{21}. One must still determine whether or not there exists a matrix X similar to X_{11} under a transformation commutative with A_1 and similar to X_{21} under a transformation commutative with A_2. If such an X exists, then the simultaneous solutions similar to X of the two equations are the transforms of X by non-singular matrices simultaneously commuta-

tive with A_1 and A_2. The problem of determining under what conditions such a matrix X exists and the nature of its transforms by matrices commutative with A_1 and A_2 is being considered by H. C. Trimble, G. Whaples, and the author.

5. **Two theorems on right linear transformations.** The following two theorems, though not necessary to the chief purpose of this paper, seem of interest. They were derived from certain suggestions of M. C. Wolf and F. A. Kiokemeister.

THEOREM 2. *The solutions of* $P(X)\odot\xi=0$ *for all vectors* ξ *are all matrices* X *for which* $h(X)=0$ *where* h *is the polynomial of maximum degree with coefficients in the centrum which divides* P.

Clearly such an X is a solution.

If $P=\sum P_i\alpha_i$, where the α's are basal elements of the division algebra D over its centrum C and the P_i have coefficients in C, then if $P(X)\odot\xi=0$ for every ξ, it follows (from Theorem 2) that $P_i(X)=0$ for every i. Hence each P_i is divisible by the minimum polynomial h_1 in C for which $h_1(X)=0$.

If $g=\sum\lambda^i a_i$ and α is an element of D, then g_α, the transform* of g, is defined to be $\sum\lambda^i\alpha^{-1}a_i\alpha$.

THEOREM 3. *A necessary and sufficient condition that* $P(X)\odot\xi$ *for every* ξ *represents a right linear transformation is that* $h(X)=0$ *where* h *is the polynomial of maximum degree with coefficients in the centrum* C *which divides* $P-P_\alpha$ *for every* α.

We have $P_\alpha(X)\odot(\xi\alpha)=(P(X)\odot\xi)\alpha$, and if $P(X)\odot\xi$ for every ξ represents a right linear transformation, then $P(X)\odot(\xi\alpha)=(P(X)\odot\xi)\alpha$ and it follows that $[P(X)-P_\alpha(X)]\xi=0$ for every ξ. Hence Theorem 3 is a consequence of Theorem 2.

6. **Example.** The following example illustrates much of the theory of this paper.

Consider the equation $X^2=A$, where

$$A=\begin{pmatrix} i & k \\ j & i-1 \end{pmatrix}.$$

It is desired to find the matrices with rational quaternionic elements which are solutions of this equation.

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* See O. Ore, *Theory of non-commutative polynomials*, Annals of Mathematics, (2), vol. 34 (1933), pp. 480–508.

The minimum equation with rational coefficients which is satisfied by A is

$$\lambda^2 + \lambda + 1 = 0,$$

and is irreducible rationally.

Hence

$$h_1 = \lambda^2 + \lambda + 1,$$

where

$$\lambda^2 + \lambda + 1 = [\lambda + (1 + i + j + k)/2][\lambda + (1 - i - j - k)/2].$$

Therefore $m_1(1) = 2$, and $\langle\langle h_1 \rangle\rangle = 1$.

Moreover,

$$h_1(\lambda^2) = (\lambda^2 + \lambda + 1)(\lambda^2 - \lambda + 1),$$

and with

$$f_1 = h_1, \qquad f_2 = \lambda^2 - \lambda + 1,$$

it follows that $l_{11} = l_{12} = 1$ and $\langle\langle f_1 \rangle\rangle = \langle\langle f_2 \rangle\rangle = 1$. Equation (2) of §3 becomes

$$n_1(1) + n_2(1) = 2$$

and has solutions

$$\begin{aligned} n_1(1) &= 2, \qquad n_2(1) = 0, \\ n_1(1) &= 0, \qquad n_2(1) = 2, \\ n_1(1) &= n_2(1) = 1. \end{aligned}$$

Hence if

$$Y_1 = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \qquad Y_2 = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix},$$

$$Y_3 = \begin{pmatrix} -(1 + i + j + k)/2 & 0 \\ 0 & (1 + i + j + k)/2 \end{pmatrix},$$

then Y_1, Y_2, and Y_3 will be similar to solutions of $X^2 = A$.

Since $T^{-1}Y_3^2T = A$ when

$$T = \begin{pmatrix} 1 & (1 + i + j + k)/2 \\ i & (1 - i - j + k)/2 \end{pmatrix},$$

it follows that

$$X_3 = T^{-1}Y_3T = \begin{pmatrix} 0 & -i \\ -1 & -j \end{pmatrix}$$

is a solution.

Similarly

$$X_1 = \begin{pmatrix} -1-i & -k \\ -j & -i \end{pmatrix}$$

and $X_2 = -X_1$ are solutions of the above equations similar to Y_1 and Y_2 respectively. All matrices commutative with A can be seen to be of the form $Q = PSP^{-1}$ where S is commutative with

$$P^{-1}AP = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$$

and

$$P = \begin{pmatrix} 1 & -1-i \\ 0 & -j \end{pmatrix}.$$

The matrix Q can be shown to be of the form

$$\begin{pmatrix} q_1 - iq_2 & q_1k - iq_1j - iq_2k - q_2j - iq_2j \\ -jq_2 & -jq_2k - jq_1j - jq_2j \end{pmatrix}.$$

This is commutative with X_1 and X_2 but not with X_3 except for particular values of the quaternions q_1 and q_2. It follows that all solutions of $X^2 = A$ are X_1, X_2, and the infinite family of matrices $Q^{-1}X_3Q$, where Q is a non-singular matrix of the type given above.

UNIVERSITY OF WISCONSIN

ON THE OPERATIONAL DETERMINATION OF TWO DIMENSIONAL GREEN'S FUNCTIONS IN THE THEORY OF HEAT CONDUCTION†

A. N. LOWAN

1. **Introduction.** In a previous paper‡ the writer has described an operational method for evaluating Green's functions in the theory of heat conduction and illustrated the method for the case of a semi-infinite solid. In this case the starting point was the solution of the differential equation of heat conduction satisfying the condition of a plane source.

It is the object of this paper to illustrate the same method for the case of the two dimensional flow of heat, in which the starting point is the solution of the differential equation of heat conduction satisfying the condition for a line source.

Specifically, we shall determine the Green's functions for the cases where the solid is one of the two following:

(A) An infinite cylinder.

(B) A solid bounded internally by a cylinder.

In both cases we shall take the boundary condition in the form

$$\frac{\partial u}{\partial r} + hu = 0 \qquad \text{for} \qquad r = a.$$

From the general solution to be derived it will be easy, by making $h=0$ or $h=\infty$ in the general solution,§ to obtain the corresponding solutions for the two important cases where the boundary is (1) impervious to heat, (2) kept at 0°.

2. **Case** (A). We start with the solution

$$(1) \qquad u(r, \theta, t; r_0, \theta_0) = \frac{1}{4\pi kt} \exp\left\{-\frac{r^2 + r_0^2 - 2rr_0 \cos(\theta - \theta_0)}{4kt}\right\}$$

which satisfies the condition for a line source at (r_0, θ_0). The solution (1) may be written in the equivalent forms

† Presented to the Society, October 30, 1937.

‡ Philosophical Magazine, (7), vol. 24 (1937), pp. 62–70.

§ Some special cases of the problems discussed in this paper have been treated by S. Goldstein, Proceedings of the London Mathematical Society, (2), vol. 34 (1932), pp. 51–88. Goldstein treats the case where the line source coincides with the axis of the cylinder. His boundary condition is $u=0$ or $\partial u/\partial r=0$.

$$(1') \qquad u(r, \theta, t; r_0, \theta_0) = \frac{1}{2\pi}\int_0^\infty \alpha e^{-k\alpha^2 t} J_0(R\alpha)d\alpha,$$

where $R^2 = r^2 + r_0^2 - 2rr_0 \cos(\theta - \theta_0)$, or

$$(1'') \quad u(r, \theta, t; r_0, \theta_0) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} \cos n(\theta - \theta_0) \cdot \int_0^\infty \alpha e^{-k\alpha^2 t} J_n(\alpha r) J_n(\alpha r_0)d\alpha.$$

The Laplace transform of (1'') is

$$u^*(r, \theta, p; r_0, \theta_0) = \int_0^\infty e^{-pt} u(r, \theta, t; r_0, \theta_0)dt,$$

where p is a complex parameter whose real part is positive. From (1'') we get

$$(2) \qquad u^*(r, \theta, p; r_0, \theta_0) = \frac{1}{2\pi k}\sum_{n=-\infty}^{\infty} \cos n(\theta - \theta_0) \cdot \int_0^\infty \frac{\alpha}{\alpha^2 - q^2} J_n(\alpha r) J_n(\alpha r_0)d\alpha,$$

where we have put $p = -kq^2$.

Consider the integral

$$(3) \qquad \begin{aligned} S &= \int_{-\infty}^{\infty} \frac{\alpha}{\alpha^2 - q^2} J_n(\alpha r) H_n^1(\alpha r_0)d\alpha \\ &= \int_{-\infty}^{0} \frac{\alpha}{\alpha^2 - q^2} J_n(\alpha r) H_n^1(\alpha r_0)d\alpha + \int_0^\infty \frac{\alpha}{\alpha^2 - q^2} J_n(\alpha r) H_n^1(\alpha r_0)d\alpha \\ &= S_1 + S_2. \end{aligned}$$

In the first term make the substitution $\alpha = -\beta$. The integral becomes

$$-\int_0^\infty \frac{\alpha}{\alpha^2 - q^2} J_n(-\alpha r) H_n^1(-\alpha r_0)d\alpha,$$

where we have used α once more for the variable of integration. But

$$J_n(-\alpha r) = (-1)^n J_n(\alpha r)$$

and

$$H_n^1(-\alpha r_0) = (-1)^n [H_n^1(\alpha r_0) - 2J_n(\alpha r_0)].$$

Hence

$$S_1 = -\int_0^\infty \frac{\alpha d\alpha}{\alpha^2 - q^2} J_n(\alpha r)[H_n^1(\alpha r_0) - 2J_n(\alpha r_0)]$$

$$= -S_2 + 2\int_0^\infty \frac{\alpha d\alpha}{\alpha^2 - q^2} J_n(\alpha r)J_n(\alpha r_0),$$

whence

$$(4) \qquad \int_0^\infty \frac{\alpha d\alpha}{\alpha^2 - q^2} J_n(\alpha r)J_n(\alpha r_0) = \frac{1}{2}\int_{-\infty}^\infty \frac{\alpha d\alpha}{\alpha^2 - q^2} J_n(\alpha r)H_n^1(\alpha r_0).$$

The last identity evidently remains valid if we interchange r and r_0.

Consider the complex integral

$$\int_C \frac{\alpha d\alpha}{\alpha^2 - q^2} J_n(\alpha r)H_n^1(\alpha r_0)$$

in the case $r<r_0$, the path of integration C consisting of the axis of reals and an infinite semicircle in the upper half of the α-plane where the path is indented by a small semicircle around the origin. The contribution to the integral tends to zero as the radius of this semicircle tends to zero. From the known asymptotic behavior of J_n and H_n^1 it is apparent that $|J_n(\alpha r)\cdot H_n^1(\alpha r_0)| \to 0$ as $|\alpha| \to \infty$. Thus the contribution to the integral from the infinite semicircle vanishes in the limit. Since the path C contains the single pole $\alpha = q$, the application of Cauchy's theorem leads at once to the identity†

$$(5) \qquad \int_0^\infty J_n(\alpha r_0)\cdot J_n(\alpha r) \frac{\alpha d\alpha}{\alpha^2 - q^2} = \frac{\pi i}{2} J_n(rq)H_n^1(r_0 q), \qquad r < r_0.$$

In an entirely similar manner we obtain the identity

$$(5') \qquad \int_0^\infty J_n(\alpha r)\cdot J_n(\alpha r_0) \frac{\alpha d\alpha}{\alpha^2 - q^2} = \frac{\pi i}{2} J_n(r_0 q)\cdot H_n^1(rq), \qquad r > r_0.$$

With the aid of the identities (5) and (5'), (2) yields

$$(6) \qquad u^*(r, \theta, p; r_0, \theta_0) = \frac{i}{4k} \sum_{n=-\infty}^{\infty} \cos n(\theta - \theta_0) J_n(r_0 q)\cdot H_n^1(rq), \qquad r > r_0,$$

and

$$(6') \qquad u^*(r, \theta, p; r_0, \theta_0) = \frac{i}{4k} \sum_{n=-\infty}^{\infty} \cos n(\theta - \theta_0)\cdot J_n(rq)H_n^1(r_0 q), \qquad r < r_0.$$

† This is one of several identities obtained by Hankel, Mathematische Annalen, vol. 8 (1875), pp. 453–470, by integrating a more general integrand around the appropriate path.

From (6) we get

$$\left(\frac{\partial u^*}{\partial r} + hu^*\right)_{r=a} = \frac{i}{4k} \sum_{n=-\infty}^{\infty} \cos n(\theta - \theta_0) \cdot J_n(r_0 q) \cdot \left\{ q \frac{d}{dz} H_n^1(z) + hH_n^1(z) \right\}_{z=aq}. \tag{7}$$

In order to obtain the Green's function we must add to the line source solution u a function v satisfying the differential equation of heat conduction and vanishing at $t=0$. Its Laplace transform must then satisfy the differential equation

$$\Delta v^* + q^2 v^* = 0, \tag{8}$$

whence

$$v^* = \frac{i}{4k} \sum_{n=-\infty}^{\infty} A_n \cos n(\theta - \theta_0) J_n(qr). \tag{9}$$

From (9) we get

$$\left(\frac{\partial v^*}{\partial r} + hv^*\right)_{r=a} = \frac{i}{4k} \sum_{n=-\infty}^{\infty} A_n \cos n(\theta - \theta_0) \left\{ q \frac{d}{dz} J_n(z) + hJ_n(z) \right\}_{z=qa}. \tag{10}$$

Since

$$\left(\frac{\partial}{\partial r} + h\right)(u^* + v^*) = 0 \quad \text{at} \quad r = a, \tag{11}$$

it follows that

$$A_n = -J_n(r_0 q) \frac{q\left[\frac{d}{dz} H_n^1(z)\right]_{z=aq} + hH_n^1(aq)}{qJ_n'(aq) + hJ_n(aq)}, \tag{12}$$

and therefore ultimately

$$u^* + v^* = \frac{i}{4k} \sum_{n=-\infty}^{\infty} \cos n(\theta - \theta_0) W_n^*, \tag{13}$$

where

$$pW_n^* = p\,\frac{J_n(qr)}{U_n(aq)}\left\{H_n^1(r_0q)U_n(aq) - J_n(r_0q)\cdot\left[\frac{z}{a}\,\frac{d}{dz}H_n^1(z) + hH_n^1(z)\right]_{z=aq}\right\} \tag{14}$$

and $U_n(aq) = qJ_n'(aq) + hJ_n(aq)$.

Remembering that $p = -kq^2$, we see that the expression pW_n^* is of the form $Y(p)/Z(p)$. The transition from pW_n^* to W_n is equivalent to the inversion of the Laplace transform defining W_n^*, and we have

$$W_n = \frac{Y(0)}{Z(0)} + \sum \frac{Y(p_i)}{p_iZ'(p_i)}\,e^{p_it}, \tag{15}$$

the summation being extended over the roots of $Z(p) = 0$. We proceed to evaluate the second member of (15). The first term is evidently zero. Further

$$Z(p) = Z(-kq^2) = U_n(aq) = qJ_n'(aq) + hJ_n(aq),$$

and therefore

$$\frac{d}{dp}Z(p) = \frac{dU_n}{dq}\,\frac{dq}{dp} = -\frac{a}{2kq}\left\{\frac{d}{dz}\left[\frac{z}{a}J_n'(z) + hJ_n(z)\right]\right\}_{z=aq} = \frac{-1}{2kq}\left\{zJ_n''(z) + (1+ah)J_n(z)\right\}_{z=aq}. \tag{16}$$

But

$$J_n''(z) + \frac{1}{z}J_n'(z) + \left(1 - \frac{n^2}{z^2}\right)J_n(z) = 0, \tag{17}$$

and from $Z(p) = U_n(aq) = 0$ we get

$$\frac{z}{a}J_n'(z) + hJ_n(z) = 0 \qquad \text{for} \qquad z = aq. \tag{18}$$

In view of (17) and (18), (16) becomes

$$Z'(p) = \frac{a}{2kq^2}\left(h^2 + q^2 - \frac{n^2}{q^2}\right)J_n(aq). \tag{16'}$$

In evaluating $Y(p_i)$ from (14) it is clear that the first term in brackets vanishes and thus

$$(19)\qquad Y(p_i) = -\,p_i J_n(q_i r) J_n(q_i r_0) \left\{ \frac{z}{a}\,\frac{d}{dz}\,H_n^1(z) + hH_n^1(z) \right\}_{z=aq_i}.$$

If we make use of the identity

$$(20)\qquad J_n(z)\frac{d}{dz}H_n^1(z) - H_n^1(z)\frac{d}{dz}J_n(z) = \frac{2i}{\pi z}$$

and of (18), the expression in braces in (19) becomes

$$\frac{2i}{\pi a J_n(q_i a)},$$

and therefore

$$(19')\qquad Y(p_i) = Y(-kq^2) = \frac{-2ip_i}{\pi a}\,\frac{J_n(rq_i)J_n(r_0 q_i)}{J_n(aq_i)}.$$

With the aid of (15), (16′), and (19′) the inversion of (13) yields

$$(21)\qquad G(r, \theta, t; r_0, \theta_0) = u + v = \frac{1}{\pi a^2}\sum_{n=-\infty}^{\infty}\cos n(\theta - \theta_0)\sum_{q_i} q_i^2 e^{-kq_i^2 t} \cdot \frac{J_n(q_i r)J_n(q_i r_0)}{\left(h^2 + q_i^2 - \dfrac{n^2}{a^2}\right)\{J_n(q_i a)\}^2},$$

where the second summation is extended over the roots of

$$(22)\qquad qJ_n'(aq) + hJ_n(aq) = 0.$$

From the general solution (22) we may obtain the solution for the case where the boundary is impervious to heat by putting $h = 0$. Also the case where the boundary is kept at 0° may be obtained by putting $h = \infty$. In this case it is clear that the transcendental equation (22) reduces to

$$(23)\qquad J_n(aq) = 0.$$

Also it is easily seen that the denominator of (22) becomes $q^2\{J_n'(q_i a)\}^2$.

Thus the Green's function for the case where the boundary is kept at 0° is

$$(24)\qquad G(r, \theta, t; r_0, \theta_0) = \frac{1}{\pi a^2}\sum_{-\infty}^{\infty}\cos n(\theta - \theta_0)\sum_{q_i} e^{-kq_i^2 t}\frac{J_n(q_i r)J_n(q_i r_0)}{\{J_n'(qa)\}^2},$$

where the second summation extends over the roots of (23).

3. **Case** (B). In this case, since the solution v^* must be finite for $r=\infty$, and since $H_n^1(z)\to 0$ in the upper half of the z-plane, it follows that we must put

$$v^*(r,\theta,p)=\frac{i}{4k}\sum_{n=-\infty}^{\infty}A_n\cos n(\theta-\theta_0)\cdot H_n^1(qr). \tag{25}$$

From (25) we get

$$\left(\frac{\partial v^*}{\partial r}+hv^*\right)_{r=a} = \frac{i}{4k}\sum A_n\cos n(\theta-\theta_0)\left\{q\frac{d}{dz}H_n^1(z)+hH_n^1(z)\right\}_{z=aq}. \tag{26}$$

Also in this case, in view of (6) and (6′), we have

$$\left(\frac{\partial u^*}{\partial r}+hu^*\right)_{r=a} = \frac{i}{4k}\sum\cos n(\theta-\theta_0)\cdot H_n^1(r_0q)\left\{q\frac{d}{dz}J_n(z)+hJ_n(z)\right\}_{z=aq}. \tag{27}$$

The condition

$$\left(\frac{\partial}{\partial r}+h\right)(u^*+v^*)=0 \qquad \text{for} \qquad r=a \tag{28}$$

thus yields ultimately

$$u^*+v^*=\frac{i}{4k}\sum_{n=-\infty}^{\infty}\cos n(\theta-\theta_0)W_n^*, \tag{29}$$

where

$$W_n^* = J_n(rq)H_n^1(r_0q) - H_n^1(r_0q)\cdot H_n^1(rq)\left\{\frac{q\frac{d}{dz}J_n(z)+hJ_n(z)}{q\frac{d}{dz}H_n^1(z)+hH_n^1(z)}\right\}_{z=aq}. \tag{30}$$

Consider first the case $h=\infty$. In this case

$$W_n^* = J_n(rq)H_n^1(r_0q)-\frac{J_n(aq)}{H_n^1(aq)}\cdot H_n^1(r_0q)\cdot H_n^1(rq) = W_{n1}^*+W_{n2}^*. \tag{31}$$

In view of (4) and (5) we have

$$W_{n1}^{*} = \frac{1}{\pi i}\int_{-\infty}^{\infty} \frac{\alpha}{\alpha^2 - q^2} J_n(\alpha r) H_n^1(\alpha r_0) d\alpha. \tag{32}$$

It will be convenient to write W_{n2}^{*} in the form of a definite integral. For this purpose consider the integral

$$\int_C \frac{\alpha}{\alpha^2 - q^2} \frac{J_n(\alpha a)}{H_n^1(a\alpha)} \cdot H_n^1(r_0\alpha) H_n^1(r\alpha) d\alpha$$

over the path C. From the asymptotic expansion of $J_n(z)$ and $H_n^1(z)$ it is easily seen that $H_n^1(r\alpha)/H_n^1(a\alpha)$ remains finite as $|\alpha| \to \infty$, and that $J_n(a\alpha)H_n^1(r_0\alpha) \to 0$ as $|\alpha| \to \infty$. Under these conditions the contribution from the large semicircle approaches zero as $|\alpha| \to \infty$. Furthermore the zeros of $H_n^1(z)$ are known to lie in the lower half of the z-plane. Thus the only pole of the integrand in the interior of C is $\alpha = q$. Cauchy's theorem thus yields

$$\begin{aligned} W_{n2}^{*} &= \frac{J_n(aq)}{H_n^1(aq)} H_n^1(rq) H_n^1(r_0 q) \\ &= \frac{1}{\pi i}\int_{-\infty}^{\infty} \frac{\alpha}{\alpha^2 - q_2} \frac{J_n(a\alpha)}{H_n^1(a\alpha)} H_n^1(r\alpha) H_n^1(r_0\alpha) d\alpha. \end{aligned} \tag{33}$$

In view of (32) and (33) we get

$$W_n^{*} = \frac{1}{\pi i}\int_{-\infty}^{\infty} \frac{\alpha}{p + k\alpha^2} \left\{ J_n(\alpha r) H_n^1(\alpha a) - J_n(\alpha a) H_n^1(\alpha r) \right\} \frac{H_n^1(\alpha r_0)}{H_n^1(\alpha a)} d\alpha,$$

and therefore, finally,

$$\begin{aligned} u + v = \frac{1}{4\pi} \sum_{-\infty}^{\infty} \cos n(\theta - \theta_0) \cdot \int_{-\infty}^{\infty} \alpha e^{-kt\alpha^2} \\ \cdot \frac{H_n^1(\alpha r_0)}{H_n^1(\alpha a)} \left\{ J_n(\alpha r) \cdot H_n^1(\alpha a) - J_n(\alpha a) H_n^1(\alpha r) \right\} d\alpha. \end{aligned}$$

This is the solution of our problem when the cylindrical surface $r = a$ is kept at 0°.

Now consider the case where h is finite. Then

$$W_{n2}^{*} = \left\{ \frac{q \dfrac{d}{dz} J_n(z) + hJ_n(z)}{q \dfrac{d}{dz} H_n^1(z) + hH_n^1(z)} \right\}_{z = aq} H_n^1(rq) H_n^1(r_0 q).$$

Consider the integral

$$\frac{1}{\pi i}\int_C \frac{\alpha}{\alpha^2 - q^2}\left\{\frac{\alpha\frac{d}{dz}J_n(z) + hJ_n(z)}{\alpha\frac{d}{dz}H_n^1(z) + hH_n^1(z)}\right\}_{z=a\alpha} H_n^1(r\alpha)H_n^1(r_0\alpha)d\alpha$$

over the path C. As before it can be shown that the contribution from the large semicircle tends to zero as $|\alpha| \to \infty$. We now assume that the zeros of $qdH_n^1(z)/dz + hH_n^1(z)$ lie in the lower half plane. The Cauchy integral theorem leads to the identity

$$\frac{1}{\pi i}\int_{-\infty}^{\infty} \frac{\alpha}{\alpha^2 - q^2}\left\{\frac{\alpha\frac{d}{dz}J_n(z) + hJ_n(z)}{\alpha\frac{d}{dz}H_n^1(z) + hH_n^1(z)}\right\}_{z=a\alpha} H_n^1(r\alpha)H_n^1(r_0\alpha)d\alpha = W_{n2}^*,$$

whence our final solution becomes

$$G = u + v = \frac{1}{4\pi}\sum_{n=-\infty}^{\infty}\cos n(\theta - \theta_0) \cdot \int_{-\infty}^{\infty} \alpha e^{-k\alpha^2 t}\frac{H_n^1(\alpha r_0)}{U_n(\alpha a)}\left\{J_n(\alpha r)U_n(\alpha a) - U_n(\alpha r)J_n(\alpha a)\right\}d\alpha,$$

where

$$U_n(\alpha a) = \left\{\alpha\frac{d}{dz}H_n^1(z) + hJ_n(z)\right\}_{z=a\alpha}.$$

For $h = \infty$ this solution yields our previous solution (21), as it should.

We have assumed above that there are no zeros of $U_n(a\alpha)$ in the interior of the path C. Briefly this can be shown by evaluating the integral

$$\frac{1}{2\pi i}\int_C \frac{\frac{d}{dz}U_n(z)}{U_n(z)}dz$$

which, as is well known, represents the number of zeros of $U_n(z)$. Using the asymptotic expansion of $H_n^1(z)$, we find that the value of the above integral is equal to zero.

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INDEX NUMBERS AND THE GENERAL ECONOMIC EQUILIBRIUM*

F. W. DRESCH

What may be called the problem of imputation, which is a central problem of economics, is that of determining how the end product of the industrial process, called consumption goods, comes to be distributed among the individuals or classes of individuals which make up the economic system. Corresponding to the classes of individuals are classes of goods, which may in particular be arranged as follows:

1. Capital or production goods (goods used for the production of other goods).
2. Primary factors, principally labor and the service of lands.
3. Consumption goods.

In what follows, by means of indices of prices and productions which have a theoretical foundation, the state of a system is described in terms of these classes. The particular problem most completely discussed is that of equilibrium under strict competition.

1. **Virtual equations of the equilibrium situation.** Let $q_1, q_2, \cdots, q_n$ be the rates of production per unit time of n commodities, the same or different, by n different producers, the commodities selling at prices $p_1, p_2, \cdots, p_n$. These goods we suppose to be arranged in the three classes 1, 2, 3, defined above. We assume first

$$q_i = q_i(q_j^i)_{j\cdot 12}, \qquad (i\cdot 13), \tag{1}$$

denoting by q_j^i the portion of q_j which is used per unit time in the production of q_i, and meaning by (1) that the amount q_i of any commodity of classes 1 or 3, which is produced per unit time, is a "technical" function of the amounts of capital goods and primary factors devoted to it; that is, q_i is a function of all the q_j^i of classes 1 and 2 for any i of classes 1 or 3. Also we assume that

$$q_i = \sum_{j\cdot 12} q_i^j, \qquad (i\cdot 12), \tag{2}$$

meaning that all of any q_i in class 1 or class 2 is portioned among the producers of goods of classes 1 and 3. It is, in fact, a requirement of the equilibrium that no commodity is being stored up at any time t in the various compartments of the system. In particular, as stated

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in (4), below, it is assumed that all the net incomes or profits are spent on consumption goods. The amounts or rates considered need not, however, be constant in time, for the equilibria may be moving equilibria.

Profits or net incomes are given by the formulas

$$\pi_i = p_i q_i - \sum_{j \cdot 12} p_j q_j^i, \qquad (i \cdot 13),$$

$$\pi_i = p_i q_i, \qquad (i \cdot 2),$$

so that, at a time t,

$$(3) \qquad d\pi_i = q_i dp_i - \sum_{j \cdot 12} q_j^i dp_j + \sum_{j \cdot 12} \left(p_i \frac{\partial q_i}{\partial q_j^i} - p_j \right) dq_j^i, \qquad (i \cdot 13),$$

in virtue of the relations (1).

Consider now the case of *strict competition*, in which the producer supposes that the prices are not under his control, and endeavors to maximize the π_i without considering changes in the p's. In that case, from (3),

$$p_i \frac{\partial q_i}{\partial q_j^i} - p_j = 0, \qquad (i \cdot 13)(j \cdot 12),$$

and these equations may be said to represent the producer's point of view and guide his use of capital and primary factors in the process of production. By means of them, equations (3) are reduced to the following:

$$(3') \qquad d\pi_i = q_i dp_i - \sum_{j \cdot 12} q_j^i dp_j, \qquad (i \cdot 13),$$

which coordinate displacements from equilibrium, at time t.

The sum of the profits π_i for the classes 1, 3 is the quantity $\Pi = \sum_{i \cdot 13} \pi_i$. By means of (3') and (2), its differential takes the form

$$(3'') \qquad d\Pi = \sum_{i \cdot 3} q_i dp_i - \sum_{i \cdot 2} q_i dp_i.$$

On the other hand, since all of these profits are spent on consumption goods, we have

$$(4) \qquad \Pi = V_3 - V_2$$

with $V_3 = \sum_{i \cdot 3} p_i q_i$, $V_2 = \sum_{i \cdot 2} p_i q_i$, by means again of (2). Hence from (3'') and (4) we have

$$(5) \qquad \sum_{i \cdot 3} p_i dq_i = \sum_{i \cdot 2} p_i dq_i.$$

In particular, if restraints are introduced in order to state an assumption that primary factors are fully employed, that is, that the q_i of the class 2 are given functions of t, we deduce the equation

$$\sum_{i\cdot 3} p_i dq_i = 0 \tag{6}$$

as the *principle of virtual displacements for our economic system.* It may be noted that these restraints do not need to be introduced prior to the equations $p_i \partial q_i/\partial q_j^i = p_j$ which represent the plans of the producers.

2. **Class indices.** Indices may now be introduced into the system according to the Divisia formulas*

$$\frac{dQ_r}{Q_r} = \frac{\sum_{i\cdot r} p_i dq_i}{\sum_{i\cdot r} p_i q_i}, \qquad \frac{dP_r}{P_r} = \frac{\sum_{i\cdot r} q_i dp_i}{\sum_{i\cdot r} q_i p_i}, \qquad r = 1, 2, 3, \tag{7}$$

in order to compare one state of the system with another, along any hypothetical path of transformation of the system (not necessarily the actual transformation in time, as with Divisia). They may be integrated along any path, to yield a function of paths; for instance,

$$Q_2(\lambda) = Q_2(\lambda_0) \exp\left[\int_{\lambda_0}^{\lambda} \left(\sum_{i\cdot 2} p_i dq_i/V_2\right)\right], \qquad V_2 = \sum_{i\cdot 2} p_i q_i,$$

in which λ is a parameter of transformation.

With these definitions, equation (6) may be written in the form

$$dQ_3 = 0; \tag{8}$$

that is, *under the hypothesis of strict competition, with primary factors fully used, the index of consumption goods is extremal with respect to virtual displacements.* Roughly speaking, consumption commodity is a maximum. In order to justify completely this last statement, it is necessary of course to consider second order differentials and insert additional minor hypotheses on the derivatives of the technical functions $q_i = q_i(q_j^i)$.

Partial indices may be introduced by the definitions

$$(dQ_3)_1 = \frac{Q_3}{V_3} \sum_{i\cdot 3} \sum_{j\cdot 1} p_i \frac{\partial q_i}{\partial q_j^i} dq_j^i,$$

* F. Divisia, *Économique Rationnelle*, Paris, 1928, p. 268. These are the only indices whose differentials are linear in the differentials of prices and quantities, such that price indices do not change when quantities alone change, and vice versa, and such that the product of price and quantity indices is an index of money value.

this being the change in Q_3 due to changes in the amounts of capital goods used, and

$$dQ_1^3 = \frac{Q_1}{V_1} \sum_{i\cdot 1} \sum_{j\cdot 3} p_i dq_j^i ,$$

which is the part of dQ_1 which goes into the manufacture of consumption goods Q_3. The quantity defined by the relation

$$\frac{\partial Q_3}{\partial Q_1^3} = \frac{(dQ_3)_1}{dQ_1^3}$$

is thus a partial rate of change, that of consumption goods index with respect to capital goods contributions. This and the other analogous partial derivatives of Q_3 and Q_1 may be shown to satisfy the equations

$$(9) \qquad \begin{aligned} &\frac{\partial Q_1}{\partial Q_1^1} = 1, && \frac{\partial Q_1}{\partial Q_2^1} = \frac{V_2}{Q_2} \Big/ \frac{V_1}{Q_1}, \\ &\frac{\partial Q_3}{\partial Q_1^3} = \frac{V_1}{Q_1} \Big/ \frac{V_3}{Q_3}, && \frac{\partial Q_3}{\partial Q_2^3} = \frac{V_2}{Q_2} \Big/ \frac{V_3}{Q_3} \end{aligned}$$

at equilibrium. By elimination we have also

$$(9') \qquad \frac{\partial Q_3}{\partial Q_1^3} \frac{\partial Q_1}{\partial Q_2^1} = \frac{\partial Q_3}{\partial Q_2^3} .$$

Since, from the definition of indices,

$$\frac{P_3 Q_3}{P_{30} Q_{30}} = \frac{V_3}{V_{30}},$$

where the subscript 0 refers to the "base" situation, we have, if we take the "base" indices as unity, $V_3/Q_3 = V_{30}P_3$. In other words, the quantities V_3/Q_3 are not prices, and in fact cannot be, since prices have dimensions which vary from quantity to quantity; nevertheless the equations bear an interesting relation to those for a simplified system based on capital, labor, and consumption goods as if they were three commodities, namely,

$$\begin{aligned} &\frac{\partial q_1}{\partial q_2^1} = \frac{p_2}{p_1}, && \frac{\partial q_3}{\partial q_1^3} = \frac{p_1}{p_3}, && \frac{\partial q_3}{\partial q_2^3} = \frac{p_2}{p_3}, \\ &\frac{\partial q_1}{\partial q_1^1} = 1, && \frac{\partial q_3}{\partial q_1^3} \frac{\partial q_1}{\partial q_2^1} = \frac{\partial q_3}{\partial q_2^3}, && dq_3 = 0, \end{aligned}$$

and justify discussion of such systems.* In particular, the problem of imputation in the simplified system may be treated by solving these equations.

It may be noted in passing that from some points of view it is desirable to replace the parts of indices like dQ_1^3 by true partial indices, for example, with the definitions

$$\frac{dQ_1^3}{Q_1^3} = \frac{1}{V_1^3} \sum_{i\cdot 1} \sum_{j\cdot 3} p_i dq_j^i, \qquad V_1^3 = \sum_{i\cdot 1} \sum_{j\cdot 3} p_i q_i^j.$$

These in fact lead to equations analogous to (9).

3. **Monopoly and limited competition.** Given that the equilibrium of strict competition satisfies the equations of virtual displacement (5), or (6) in the case of given amounts of primary factors, one would expect an equilibrium in which exist limited forms of competition or monopoly to be expressed in a modified form of these virtual equations. In fact the latter forms of production are possible if the producer makes some sort of hypothesis about the relation of demand for his product to price, since he is not assuming that the price is outside the scope of his manipulation. In particular, the monopolist assumes that he knows the "elasticity of demand," e_i, in the relation

$$p_i dq_i = -e_i q_i dp_i, \qquad (i\cdot 13), \tag{10}$$

instead of assuming that $dp_i = 0$ when he plans to maximize his profit. This assumption leads to the equations

$$p_i \frac{\partial q_i}{\partial q_j} = r_i p_j, \qquad r_i = \frac{e_i}{e_i - 1}, \qquad (i\cdot 13)(j\cdot 12), \tag{10'}$$

which govern the planning of the monopolist in regard to the relation between his own price and production, provided that he still considers the prices involved in his cost function as given. Thus the portion of $d\pi_i$ which does not involve dp_j, $(j\cdot 12)$, is made to vanish, and

$$p_i dq_i + q_i dp_i - \sum_{j\cdot 12} p_j dq_j^i = 0, \qquad (i\cdot 13),$$

which with (10) yields (10'). By summation, we have

* G. C. Evans, *Maximum production studied in a simplified economic system*, Econometrica, vol. 2 (1934), pp. 37–50.

O. Lange, *The place of interest in the theory of production*, Review of Economic Studies, vol. 3 (1936), pp. 159–192. The equations involving p_2 for the simplified system were first published by Lange, but were used independently in The University of California seminar on mathematical economics.

$$\sum_{i\cdot 12} (p_i dq_i + q_i dp_i) = \sum_{j\cdot 12} p_i dq_i. \tag{11}$$

If the assumption of the monopolist is correct, the equations (10) may be taken as additional restraints on the system, and (11) becomes then a principle of virtual displacements. It contains (5) formally, as is seen by putting the r_i equal to unity. A similar modification occurs also in the equations (9). Thus the coefficient R_w^u, as different from unity, in equations of the type

$$\frac{\partial Q_u}{\partial Q_w^u} = R_w^u \frac{V_w}{Q_w} \Big/ \frac{V_u}{Q_u}$$

measures the departure from strict competition. The u, w may refer to subclasses of the main classes 1, 2, 3.

The relations so far treated have been obtained without the necessity of a full determination of the coordinates q_i^j, p_i, as a count of the equations will show. To complete the picture of the equilibrium situation, demand relations in general may be given, say the statistically determined fraction f_{ij} of π_j which is spent on q_i, $(i\cdot 3)$.

4. **Equilibrium theory of interest.** As an application of the theory consider the problem of the rate of interest in a situation of strict competition under equilibrium conditions. Although the interesting problem should be dynamic rather than static, we can think of the phenomena in the equilibrium case as due to the necessity of using money in production, money being owned by a special class, as if it were a primary factor like land and labor. The price of renting it, which is the rate of interest, will appear in the same category as the rent of land and the wages of labor in class 2.

But it is also convenient to introduce the rate of interest, with the same ultimate result, in terms of restraints on the system, on postulating that the total cost of production is accepted by each producer as limited to a certain amount; that is,

$$\sum_{j\cdot 12} p_j q_j^i = K_i, \qquad (i\cdot 13). \tag{12}$$

In this case, we maximize, with respect to choices of the q_j^i, the expressions

$$I_i = p_i q_i - \sum_{j\cdot 12} p_j q_j^{\,i} = p_i q_i - \sum_{j\cdot 12} p_j q_j^{\,i} + \lambda_i \Big[K_i - \sum_{j\cdot 12} p_j q_j^{\,i} \Big],$$

the restraints entering through the Lagrangian multipliers λ_i. Thus we obtain the equations

$$p_i \frac{\partial q_i}{\partial q_j^i} - (1 + \lambda_i) p_j = 0, \qquad (i \cdot 13)\ (j \cdot 12), \tag{13}$$

with the hypothesis of strict competition. Here the λ_i are not necessarily elements of cost, but the equilibrium conditions (12) are the same as if they were such elements, appearing as rates of interest. We should then define the profits by the formulas

$$\pi_i = I_i - \lambda_i K_i,$$

and as a first approximation we might assume the λ_i all equal, $\lambda_i = \lambda$.

The same formal result may be obtained by making extremal (at any time, $t = t_0$) the quantity

$$I = \sum_{i \cdot 13} p_i q_i - \sum_{j \cdot 12} p_j q_j$$

subject to the restraints

$$\sum_{j \cdot 12} p_j q_j = K = K(t_0),$$

with all the dp_i set equal to zero; in fact, there appears in this way a single λ. Between I and the value $\Pi = \sum_{i \cdot 13} \pi_i$ there holds the relation $I = \Pi + \lambda K$.

In terms of the indices of prices and quantities, the equation (9) becomes

$$\frac{V_u}{Q_u} \frac{\partial Q_u}{\partial Q_w^u} = (1 + \lambda) \frac{V_w}{Q_w},$$

so that for the three fundamental categories, with the various V_u, V_w eliminated, we have the equations

$$\frac{\partial Q_1}{\partial Q_1^1} = 1 + \lambda,$$

$$\lambda = \frac{\dfrac{\partial Q_2}{\partial Q_1^2} \dfrac{\partial Q_1}{\partial Q_2^1} - \dfrac{\partial Q_3}{\partial Q_2^3}}{\dfrac{\partial Q_3}{\partial Q_2^3}}.$$

The second of these equations states that $1 + \lambda$ is equal to the ratio of the marginal productivity of indirect factors of production (that is, productive of consumption goods indirectly through the production of capital goods) to the marginal productivity of direct factors of production. The equation is essentially the formula for the rate of

interest given by Wicksell.* It was obtained for the simplified system already mentioned, by Lange.† The remarks in this section are in fact a generalization or justification of this latter theory.

The index relations are particularly interesting in discussing changes from one system to another consequent on the introduction or change of interest rate λ. Thus, with the index of primary factors given, that is, Q_2 given, the introduction of a small interest rate λ induces no modification of Q_3 as far as differentials of the first order. In fact,

$$\delta Q_3 = 0,$$

and

$$\delta^2 Q_3 = \lambda \frac{\partial Q_3}{\partial Q_1^2} \delta Q_1.$$

Equations such as these are important for economic theory.

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NOTE ON ALMOST-UNIVERSAL FORMS‡

P. R. HALMOS

Ramanujan§ and Dickson‖ proved that there are 54 universal forms $ax^2+by^2+cz^2+dt^2$ with positive integral coefficients a, b, c, d. It is the purpose of this note to investigate *almost-universal forms*, that is, to exhibit sets of positive integral coefficients a, b, c, d such that $ax^2+by^2+cz^2+dt^2$ represents every positive integer with exactly one exception.

Ramanujan§ showed that a necessary and sufficient condition that a form $ax^2+by^2+cz^2+dt^2$ be universal is that it represent the first fifteen positive integers. Consequently the integer which an almost-universal form fails to represent cannot be greater than 15. Using Ramanujan's method of bounding the coefficients we can exhibit, merely by requiring that a form fail to represent exactly one of the

* Wicksell, *Lectures on Political Economy*, London, 1935 (translation), vol. 1, p. 156.

† Lange, loc. cit.

‡ Presented to the Society, December 28, 1937.

§ Proceedings of the Cambridge Philosophical Society, vol. 19 (1917), pp. 11–21; *Collected Papers*, Cambridge, 1927, pp. 169–178.

‖ This Bulletin, vol. 33 (1927), pp. 63–70.

first fifteen positive integers, a set of 135 forms which has to contain all almost-universal forms. The well known theories of special ternary quadratic forms,* or even empirical verification, will reduce this number to 88. (Empirical verification would sometimes be cumbersome; for example, the first integer, other than 10, that $x^2+2y^2+5z^2+15t^2$ fails to represent is 250.)

The following list exhibits the 88 possibilities for almost-universal forms (where (a, b, c, d) denotes the form $ax^2+by^2+cz^2+dt^2$):

Forms that do not represent 1:
(1)–(3) (2, 2, 3, 4), (2, 3, 4, 5), (2, 3, 4, 8).
Forms that do not represent 2:
(4)–(5) (1, 3, 3, 5), (1, 3, 5, 6).
Forms that do not represent 3:
(6)–(7) $(1, 1, 4, d)$, $d=5, 6$;
(8)–(11) $(1, 1, 5, d)$, $d=5, 6, 10, 11$;
(12)–(15) $(1, 1, 6, d)$, $d=7, 8, 10, 11$.
Forms that do not represent 5:
(16)–(20) $(1, 2, 6, d)$, $d=6, 10, 11, 12, 13$;
(21)–(25) $(1, 2, 7, d)$, $d=8, 10, 11, 12, 13$.
Forms that do not represent 6:
(26)–(32) $(1, 1, 3, d)$, $d=7, 8, 10, 11, 13, 14, 15$.
Forms that do not represent 7:
(33)–(37) $(1, 1, 1, d)$, $d=9, 10, 12, 14, 15$;
(38)–(42) $(1, 2, 2, d)$, $d=9, 10, 12, 14, 15$.
Forms that do not represent 10:
(43)–(55) $(1, 2, 3, d)$, $d=11, 12, 13, 15, 17, 19, 20, 21, 22, 23, 24, 25, 26$;
(56)–(59) $(1, 2, 5, d)$, $d=11, 12, 13, 14$.
Forms that do not represent 14:
(60)–(73) $(1, 1, 2, d)$, $d=15, 17, 18, 19, 20, 21, 22, 23, 24, 25, 27, 28, 29, 30$;
(74)–(87) $(1, 2, 4, d)$, $d=15, 17, 18, 19, 20, 21, 22, 23, 24, 25, 27, 28, 29, 30$.
Forms that do not represent 15:
(88) (1, 2, 5, 5).

One general method of proof serves to establish almost-universality for most of these forms. By way of illustrating this method, we prove the following typical theorem:

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* The properties of every ternary form needed in this note are either discussed by Dickson, loc. cit., or else they follow from Dickson's discussion by elementary means.

THEOREM 1. *The form* $2x^2+2y^2+3z^2+4t^2$ *represents every positive integer with the exception of unity.*

Since (1, 1, 2) represents all odd numbers* (where (a, b, c) denotes the ternary quadratic form $ax^2+by^2+cz^2$), (2, 2, 3, 4) represents all numbers of the form $4k+2$ with $z=0$. For $k>0$ we have $4k+1=4(k-1)+2+3$, hence (2, 2, 3, 4) represents $4k+1$ with $z=1$. Since (1, 1, 6, 2) is a universal form,† for every $k\geqq 0$ we have $2k=x^2+y^2+6z^2+2t^2$, whence $4k=2x^2+2y^2+3(2z)^2+4t^2$. Finally, it may be proved, by consideration of elementary divisibility properties, that two numbers not represented by (1, 1, 2) never differ by 12. Hence, for $k>5$ we have either $2k=x^2+y^2+2t^2$ or else $2k-12=x^2+y^2+2t^2$. According as the first or the second relation holds we have $4k+3=2x^2+2y^2+3\cdot 1^2+4t^2$ or else $4k+3=2x^2+2y^2+3\cdot 3^2+4t^2$. Since it is readily verified that (2, 2, 3, 4) represents the numbers $4k+3$, $k=0, 1, 2, 3, 4, 5$, the proof of the theorem is complete.

The above treatment is not applicable to the form (2): (2, 3, 4, 5). We prove the following theorem:

THEOREM 2. *The form* $2x^2+3y^2+4z^2+5t^2$ *represents every positive integer with the exception of unity.*

Let A, B, C be three numbers of the form $4^\alpha(16k+10)$, where α and k are non-negative integers. Concerning these we have the following lemma:

LEMMA. *It is impossible that the two equations* $A-B=40, A-C=120$ *hold simultaneously; also, the equation* $A-B=20$ *is impossible.*

The proof of the lemma is elementary and is omitted.

Empirical verification yields the result that the form (2, 3, 4, 5) represents all integers n where $2\leqq n\leqq 200$.

Consider now the ternary form (1, 2, 6). It represents every positive integer not of the form $4^\alpha(8k+5)$. But if $n=x^2+6y^2+2z^2$, then $2n=2x^2+3(2y)^2+4z^2$, whence the form $2x^2+3y^2+4z^2+5t^2$ represents all even numbers not of the form $4^\alpha(16k+10)$, with $t=0$. But if A is such a number, then, by the lemma, $A-20$ is not; whence $A-20=2x^2+3y^2+4z^2$, and $A=2x^2+3y^2+4z^2+5\cdot 2^2$. Hence the form (2, 3, 4, 5) represents all even numbers. It also represents all odd numbers, excepting unity and those of the form $A+5$, with $t=1$. By the lemma, one of the even numbers $A-40$ or $A-120$ is repre-

* This Bulletin, vol. 33 (1927), pp. 63–70.

† Ramanujan, loc. cit.

sented by (2, 3, 4); whence $A+5$ is represented by (2, 3, 4, 5) with $t=3$ or $t=5$ respectively. This completes the proof of the theorem.

One of the ternary forms, namely (2, 2, 4), involved in (2, 2, 3, 4) is regular; that is, the total set of numbers which it fails to represent coincides with the total set of numbers in a certain collection of arithmetic progressions. It is this property that makes the difference between (2, 2, 3, 4) and (2, 3, 4, 5). Every ternary form involved in (2, 3, 4, 5) is irregular.* (A form f is said to be irregular if there exists a positive integer k not represented by f, but having the property that every arithmetic progression containing k contains also numbers represented by f.) We were able to prove Theorem 1, however, because two of the coefficients of the form (2, 3, 4) are not relatively prime. If the form represents a multiple of the common divisor, it becomes a multiple of the regular form (1, 2, 6).

Either the method of Theorem 1 or else that of Theorem 2 proves the almost-universality of 86 of the 88 forms. The author has not hitherto found out whether or not the two forms (23): (1, 2, 7, 11) and (25): (1, 2, 7, 13) are almost-universal in the sense of this note. Every ternary form involved in either of the two quarternary forms is irregular, and no reduction of the sort described above is possible. Each of these forms fails to represent only one positive integer $n \leqq 300$.

Professor Carmichael has recently communicated to me the following result (for the proof of which he had to employ the Dirichlet method of dealing with ternary forms): *The form* (1, 2, 11) *represents every even number not of the form* $4(16n-10)$. With the aid of this result, one may prove by the methods exhibited above that the form (1, 2, 7, 11) is almost-universal.†

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* L. E. Dickson, Annals of Mathematics, (2), vol. 28 (1927), pp. 333–341. The results of this paper are not applicable to any of the ternary forms involved in (2, 3, 4, 5), but the methods are sufficiently general to prove the assertion above.

† The last paragraph was added in proof, January 17, 1938.

METRIC SPACES WITH GEODESIC RICCI CURVES. I

JACK LEVINE

1. **Introduction.** The problem of determining all Riemannian spaces of three dimensions admitting geodesic Ricci curves has been solved by G. Ricci* and P. Walberer† using, however, different methods. Although they obtained all such V_3, the complete explicit determination of all such V_n for $n>3$ does not seem possible because of the increased number and complexity of the differential equations which arise.

In this paper the following two problems related to the above problem will be considered.

In the first problem we suppose given a set of linearly independent vectors‡ $\lambda^i_{a|}$ and wish to determine necessary and sufficient conditions on the $\lambda^i_{a|}$ in order that a set of scalars $\theta_a(\neq 0)$ exist which will define a metric space V_n with a metric determined by

$$g^{ij} = \sum_h e_h \bar{\lambda}^i_{h|} \bar{\lambda}^j_{h|}, \tag{1}$$

where

$$\bar{\lambda}^i_{a|} = \theta_a \lambda^i_{a|}, \tag{2}$$

and e_h $(=\pm 1)$ are arbitrary; and such that the congruences of curves defined by the $\lambda^i_{a|}$ will be geodesics in the V_n thus determined. (The vectors $\bar{\lambda}^i_{a|}$ define the same congruences as do the $\lambda^i_{a|}$, and these congruences form an orthogonal ennuple in the V_n.)

In the second problem we assume that these conditions on the $\lambda^i_{a|}$ have been determined and that the n congruences defined by a set of $\lambda^i_{a|}$ are geodesics in the V_n determined by

$$g^{ij} = \sum_h e_h \lambda^i_{h|} \lambda^j_{h|};$$

we then find necessary and sufficient conditions that, with respect to the metric (1), the congruences be geodesic Ricci curves.

* G. Ricci, *Sulle varieta a tre dimensioni dotate die terne principali di congruenze geodetiche*, Rendiconti della Reale Accademia dei Lincei, (5), vol. 27 (1918), pp. 21–28, 75–87.

† P. Walberer, *Riemannsche Raume mit geodätischen Riccikurven*, Hamburger Abhandlungen, vol. 10 (1934), pp. 152–168.

‡ All indices take the values $1, 2, \cdots, n$ unless otherwise noted.

The method of Walberer is followed, and we obtain generalizations to n dimensions of his conditions for 3 dimensions.*

2. **Geodesic congruences.** In this section we solve the first of the two problems stated in the introduction.

An orthogonal ennuple of unit vectors, $\lambda^i_{a|}$, of a V_n will determine geodesic congruences if†

$$\gamma_{iaa} = 0, \quad a \text{ not summed}, \tag{3}$$

where

$$\gamma_{lhk} \equiv \lambda_{l|i,j}\overset{i}{\lambda}_{h|}\overset{j}{\lambda}_{k|},$$

are the Ricci coefficients of rotation.‡

The congruences are determined from the equations

$$\frac{dx^1}{\lambda^1_{a|}} = \frac{dx^2}{\lambda^2_{a|}} = \cdots = \frac{dx^n}{\lambda^n_{a|}}, \tag{4}$$

and are defined by the $\lambda^i_{a|}$ to within a scalar factor, that is, the $\bar{\lambda}^i_{a|}$ given by (2) will define the same congruences as will the $\lambda^i_{a|}$. If then we wish to determine our conditions on the $\lambda^i_{a|}$ which make these congruences geodesics with respect to (1) we form equations (3) with $\lambda^i_{a|}$ replaced by $\bar{\lambda}^i_{a|}$ given by (2). The resulting equations in θ_a as unknowns must have solutions and this requirement leads to the desired conditions on the $\lambda^i_{a|}$.

We begin by replacing (3) by more suitable equations. If the operators Δ_a are defined by

$$\Delta_a \equiv \overset{i}{\lambda}_{a|}\frac{\partial}{\partial x^i},$$

their integrability conditions take the form§

$$(\Delta_a, \Delta_b) \equiv \Delta_a\Delta_b - \Delta_b\Delta_a = \sum_i e_i(\gamma_{iab} - \gamma_{iba})\Delta_i.$$

We write this in the form

$$(\Delta_a, \Delta_b) = \overset{k}{c}_{ab}\Delta_k,$$

* New conditions are also obtained which do not appear for $n=3$.

† L. P. Eisenhart, *Riemannian Geometry*, p. 100. References to this book will be in the form RG.

‡ RG, pp. 97–98.

§ RG, p. 99.

so that

$$c_{ij}^{k} = e_k(\gamma_{kij} - \gamma_{kji}) = -c_{ji}^{k}.$$

These equations can be solved to give

$$\gamma_{ijk} = \tfrac{1}{2}(e_i c_{jk}^{i} + e_j c_{ki}^{j} + e_k c_{ji}^{k}). \tag{5}$$

In terms of the c's the conditions (3) for geodesics become

$$c_{ij}^{j} = 0, \quad j \text{ not summed}. \tag{6}$$

We also remark, for later use, that the Jacobi identities

$$(\Delta_i, (\Delta_j, \Delta_k)) + (\Delta_j, (\Delta_k, \Delta_i)) + (\Delta_k, (\Delta_i, \Delta_j)) = 0$$

are equivalent to the conditions

$$\Delta_i c_{jk}^{l} + \Delta_j c_{ki}^{l} + \Delta_k c_{ij}^{l} = -(c_{jk}^{h} c_{ih}^{l} + c_{ki}^{h} c_{jh}^{l} + c_{ij}^{h} c_{kh}^{l}). \tag{7}$$

Writing

$$\bar{\Delta}_a = \theta_a \Delta_a,$$

we have

$$(\bar{\Delta}_a, \bar{\Delta}_b) = \bar{c}_{ab}^{k} \bar{\Delta}_k,$$

from which we obtain

$$\bar{c}_{ij}^{k} = \frac{\theta_i \theta_j}{\theta_k} c_{ij}^{k}, \qquad k \neq i, j, \tag{8}$$

$$\bar{c}_{ij}^{i} = \theta_j (c_{ij}^{i} - \mu_{ij}), \qquad i \neq j, \tag{9}$$

where

$$\mu_i = \log \theta_i, \qquad \mu_{ij} = \Delta_j \mu_i.$$

The conditions (6) for geodesics in the barred quantities are $\bar{c}_{ij}^{j} = 0$, (j not summed), and from (9) these conditions become

$$\mu_{ij} = \Delta_j \mu_i = c_{ij}^{i}, \qquad i \neq j. \tag{10}$$

These equations in the θ_a as unknowns must have solutions* and their integrability conditions give us our conditions on the $\lambda_{a|}^{t}$.

Forming the integrability conditions for (10), we obtain

* Of course, we do not assume $c_{ij}^{i} = 0$, (i not summed), as this condition need only be satisfied by $\bar{c}_{ij}^{i}$.

$$(\Delta_i, \Delta_j)\mu_k = \Delta_i c^k_{kj} - \Delta_j c^k_{ki} = c^h_{ij}\Delta_h\mu_k, \qquad k \neq i, j,$$

or

$$c^k_{ij}\Delta_k\mu_k = \Delta_i c^k_{kj} - \Delta_j c^k_{ki} - c^h_{ij}c^k_{kh}, \qquad k \neq i, j;\ k \text{ not summed.} \tag{11}$$

If in (7) we put $l = k$ (without summing), and transpose appropriate terms, we obtain

$$\Delta_i c^k_{kj} - \Delta_j c^k_{ki} - c^h_{ij}c^k_{kh} = \Delta_k c^k_{ij} + c^h_{jk}c^k_{ih} + c^h_{ki}c^k_{jh}, \qquad k \text{ not summed.}$$

Substitution from this equation into (11) gives

$$c^k_{ij}\Delta_k\mu_k = \Delta_k c^k_{ij} + c^h_{jk}c^k_{ih} + c^h_{ki}c^k_{jh}, \qquad k \neq i, j;\ k \text{ not summed.} \tag{12}$$

The equations (12) are the desired integrability conditions of (10).

If $c^k_{ij} = 0$, $(k \neq i, j)$, for every k, equations (12) are satisfied identically. We thus assume that for each k there is at least one $c^k_{ij} \neq 0$. We can then write (12) in the form*

$$\Delta_k\mu_k = \frac{1}{c^k_{ij}}(\Delta_k c^k_{ij} + c^h_{jk}c^k_{ih} + c^h_{ki}c^k_{jh}), \qquad k \neq i, j, \tag{13}$$

and equations (13) are to hold for all i, j such that $c^k_{ij} \neq 0$.

Since the left member of (13) is independent of i, j, so must be the right member, that is, we must have

$$\frac{1}{c^k_{ij}}(\Delta_k c^k_{ij} + c^h_{jk}c^k_{ih} + c^h_{ki}c^k_{jh}) = \frac{1}{c^k_{ab}}(\Delta_k c^k_{ab} + c^h_{bk}c^k_{ah} + c^h_{ka}c^k_{bh}) \equiv H^k. \tag{14}$$

Equations (14) are to hold for all i, j, a, b such that

$$c^k_{ij} \neq 0, \qquad c^k_{ab} \neq 0, \qquad k \neq i, j, a, b.$$

We now consider (13) and (10) combined, and form their integrability conditions. The only new conditions are obtained from $(\Delta_l, \Delta_k)\mu_k$, these conditions reducing to

$$H^{kl} \equiv \Delta_l H^k - \Delta_k c^k_{kl} - c^h_{lk}c^k_{kh} + c^k_{kl}H^k = 0, \qquad k \neq l. \tag{15}$$

If for given k, i, j, $(k \neq i, j)$, $c^k_{ij} = 0$, equations (12) reduce to

$$H^k_{ij} \equiv c^h_{jk}c^k_{ih} + c^h_{ki}c^k_{jh} = 0, \tag{16}$$

which must be satisfied for all such k, i, j.

* If for any given k, $c^k_{ij} = 0$, for all i and j, the corresponding equation is identically zero and may be dropped.

Under (8), (9), we find that the various H's transform in the following manner:

$$\overline{H}^{k} = \theta_k(H^{k} - \mu_{kk}), \qquad \overline{H}^{k}_{(ij)(ab)} = \theta_k H^{k}_{(ij)(ab)},$$
$$\overline{H}^{kl} = \theta_k\theta_l H^{kl}, \qquad \overline{H}^{k}_{ij} = \theta_k\theta_j H^{k}_{ij}.$$

The quantities $H^{k}_{(ij)(ab)}$ are the result of transposing the right member of (14) to the left, giving $H^{k}_{(ij)(ab)} = 0$.

The quantities H^{kl} are the generalizations to n dimensions of the corresponding quantities for $n = 3$ given by Walberer. The other H's do not appear for $n = 3$.

We can state the following theorem:

Given a set of independent vectors $\lambda^{i}_{a|}$, necessary and sufficient conditions that a set of scalars θ_a should exist such that the vectors $\overline{\lambda}^{i}_{a|} = \theta_a\lambda^{i}_{a|}$ determine an orthogonal ennuple of geodesic congruences in the metric space defined by

$$g^{ij} = \sum_{k} e_k \overline{\lambda}^{i}_{k|}\overline{\lambda}^{j}_{k|}, \tag{17}$$

are

(a) $$H^{kl} = 0, \qquad k \neq l,$$

(b) $$H^{k}_{(ij)(ab)} = 0, \qquad k \neq i, j, a, b,$$

(c) $$H^{k}_{ij} = 0, \qquad k \neq i, j.$$

Conditions (a) *are used for all indices k for which H^{k} is defined. Conditions* (b) *are used only for indices $i, j, a, b, (\neq k)$, such that $c^{k}_{ij} = 0$, $c^{k}_{ab} = 0$. Conditions* (c) *are used only for indices k, i, j such that $c^{k}_{ij} = 0$, $(k \neq i, j)$.*

It is to be noted that if $c_{ij}{}^{i} = 0$, (i not summed), these integrability conditions are satisfied identically (as is to be expected). In this case $H^{k} = 0$, as is seen from (7) on putting $k = l$. Then (a) and (b) follow, and (c) follows from the Jacobi identities (7).

From the definitions of the H's it is seen that the conditions (a), (b), (c) are independent of the e_k.

3. **Geodesic Ricci curves.** We consider now the second of the two problems stated in the introduction. We suppose given a set of $\lambda^{i}_{a|}$ satisfying conditions (a), (b), (c) of section 2. Then there exists a set of the θ_a such that $\bar{c}^{i}_{ij} = 0$, (i not summed), and thus we obtain the $\overline{\lambda}^{i}_{a|}$ of (2). We now drop the bars and suppose given a set of λ's such

that $c_{ij}^{i}=0$, (i not summed), and wish to find a set of scalars θ_a ($\neq 0$) such that $\bar{\lambda}_{a|}^{i}=\theta_a\lambda_{a|}^{i}$ will determine the geodesic Ricci curves of a V_n defined by (1).

An orthogonal ennuple of unit vectors, $\lambda_{a|}^{i}$, of a V_n will define the n Ricci congruences of the V_n if

$$R_{ij}\lambda_{a|}^{i}\lambda_{b|}^{j}=0, \qquad a\neq b, \tag{18}$$

where R_{ij} are the components of the Ricci tensor.*

From

$$\begin{aligned}\gamma_{lpqr} \equiv R_{hijk}\lambda_{l|}^{h}\lambda_{p|}^{i}\lambda_{q|}^{j}\lambda_{r|}^{k} &= \Delta_r\gamma_{lpq}-\Delta_q\gamma_{lpr}\\ &+\sum_m e_m[\gamma_{lpm}(\gamma_{mqr}-\gamma_{mrq})+\gamma_{mlr}\gamma_{mpq}-\gamma_{mlq}\gamma_{mpr}],\end{aligned} \tag{19}$$

and

$$R_{ij}=g^{hk}R_{hijk},$$

we obtain as conditions equivalent to (18),

$$\sum_a e_a\gamma_{abca}=0, \qquad b\neq c. \tag{20}$$

If we assume (3) satisfied (condition for geodesics), conditions (20) reduce to

$$\sum_{a,m} e_a(\Delta_a\gamma_{abc}+e_m\gamma_{abm}\gamma_{mca})=0, \qquad b\neq c. \tag{21}$$

If in (7) we put $l=k$, and sum, and use (6), we obtain

$$\Delta_k c_{ij}^{k}=0. \tag{22}$$

If now we substitute from (5) in (21) and make use of (6) and (22) we eventually reduce (21) to the form

$$\begin{aligned}&\sum_a e_a(e_b\Delta_a c_{ac}^{b}+e_c\Delta_a c_{ab}^{c})\\ &\quad=\sum_{a,m} e_ae_m[e_ae_m c_{bm}^{a}c_{ca}^{m}+c_{bm}^{a}c_{cm}^{a}-\tfrac{1}{2}e_be_c c_{am}^{b}c_{am}^{c}],\end{aligned} \qquad b\neq c. \tag{23}$$

We must impose conditions (23) and (6) in the $\bar{c}$ to obtain geodesic Ricci curves.

From (10) we now have

$$\mu_{ij}=\Delta_j\mu_i=0, \qquad i\neq j. \tag{24}$$

* RG, pp. 110, 114.

Hence,

$$(\Delta_i, \Delta_j)\mu_k = \Delta_i\mu_{kj} - \Delta_j\mu_{ki} = 0 = \overset{m}{c}_{ij}\Delta_m\mu_k = \overset{k}{c}_{ij}\Delta_k\mu_k, \qquad k \neq i, j,$$

or

$$\overset{k}{c}_{ij}\mu_{kk} = 0, \qquad k \neq i, j;\ k \text{ not summed.} \tag{25}$$

The Jacobi conditions in the $\bar{c}$ add no new conditions on the θ_a, for from (7) in the $\bar{c}$ we get (24) and (25) again.

We now form (23) in the $\bar{c}$ and thus get the remaining conditions on the θ. We find on forming (23) in the $\bar{c}$ and then substituting from (8) and (9) that the resulting conditions can be written in the form

$$\begin{aligned}\sum_a e_a\theta_a^2[(e_b\theta_c^2\overset{b}{c}_{ac} + e_c\theta_b^2\overset{c}{c}_{ab})\mu_{aa} + e_b\theta_c^2\Delta_a\overset{b}{c}_{ac} + e_c\theta_b^2\Delta_a\overset{c}{c}_{ab}] \\ = \sum_{a,m} e_a e_m\left[e_a e_m\theta_b^2\theta_c^2\overset{a}{c}_{bm}\overset{m}{c}_{ca} + \frac{\theta_1^2\theta_2^2\theta_3^2}{\theta_a^2}\overset{a}{c}_{bm}\overset{a}{c}_{cm} - \tfrac{1}{2}e_b e_c\theta_a^2\theta_m^2\overset{b}{c}_{am}\overset{c}{c}_{am}\right],\end{aligned} \tag{26}$$

$$b \neq c.$$

The conditions on the θ_i are given by (24), (25), and (26). We shall solve the simplest case here, that is, the one in which all the c's are zero but one, which we may assume to be $\overset{1}{c}_{23}$. (If all the c's are zero, the space is flat.)

From (25) we see $\mu_{11} = 0$, which together with (24) gives

$$\theta_1 = k_1 = \text{const.}$$

The only conditions obtained from (26) are

$$\mu_{33} + \Delta_2 A = 0, \tag{27}$$

$$\mu_{22} + \Delta_2 A = 0, \tag{28}$$

where

$$\overset{1}{c}_{23} = a, \qquad A = \log a.$$

Since now

$$(\Delta_i, \Delta_j) = \overset{1}{c}_{ij}\Delta_1,$$

we have

$$(\Delta_i, \Delta_j) = 0, \qquad i, j \neq 2.$$

Hence the operators Δ_i, $(i \neq 2)$, define an abelian group,* and by a change of coordinate system, we have the canonical form

* L. P. Eisenhart, *Continuous Groups of Transformations*, p. 49.

$$\Delta_i = \frac{\partial}{\partial x^i}, \qquad i \neq 2, \tag{29}$$

$$\Delta_2 = \alpha(x^2, x^3)\frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2}, \tag{30}$$

and

$$a = -\frac{\partial \alpha}{\partial x^3}.$$

Equations (24), (29), (30) now show that

$$\mu_i = \mu_i(x^i), \qquad \text{or} \qquad \theta_i = \theta_i(x^i),$$

and then from (27) and (28) we obtain $\partial^2 A/\partial x^2 \partial x^3 = 0$. This gives

$$a = B(x^2)C(x^3),\ \mu_2' = -\frac{B'}{B},\ \mu_3' = -\frac{C'}{C},\ \alpha = -BD,\ D = \int C dx^3,$$

or

$$\theta_2 = \frac{k_2}{B}, \qquad \theta_3 = \frac{k_3}{C}.$$

Then $\hat{c}^1_{23} = k$ (=const.), and the $\bar{\Delta}_a$, by means of the coordinate transformation

$$x'^1 = \frac{x^1}{k_1},\quad x'^2 = \int \frac{B}{k_2} dx^2,\quad x'^3 = \frac{D}{k_3},\quad x'^i = \int \frac{dx^i}{\theta^i}, \qquad i > 3,$$

can be reduced to the canonical form (by dropping primes)

$$\bar{\Delta}_i = \frac{\partial}{\partial x^i}, \qquad i \neq 2,$$

$$\bar{\Delta}_2 = -kx^3\frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2}.$$

From (17) we have

$g^{ij} = e_i \delta^j_j$, $(i, j \neq 1, 2)$; $g^{11} = e_1 + e_2(kx^3)^2$, $g^{12} = -ke_2x^3$, $g^{22} = e_2$, from which the g_{ij} are easily obtained.

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